PATTERNS IN ENVIRONMENTAL MATHEMATICS

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NATURE-LOVER.NET/MATH

GROWTH : GEOMETRIC SEQUENCES AND EXPONENTIAL FUNCTIONS

2/6/2019 CLASS NOTES

TEST COMING UP

MARK YOUR CALENDARS!

FIRST TEST WILL BE ON FEB 22, FRIDAY

REVIEW IN CLASS ON FEB 20 WEDNESDAY

WILL COVER LINEAR, QUADRATIC AND EXPONENTIAL FUNCTIONS (AND LOGARITHMS, DEPENDING ON WHETHER WE GET THAT FAR)

Problem from College Algebra

- Suppose you are offered a prize and two ways of receiving it:
- Either you get a million dollars at the end of the month
- Or you get one dollar on first day, two dollars on second, 4 dollars on third,…and so on, the amount doubling on each day.
- Which one would you accept?

How much would you have if you accept the doubling method?

- Say that month had 30 days.
- You get \$ 1 on the 1st, \$ 2 on the 2nd, \$ 4 on the 3rd, \$ 8 on the fourth, and so on.
- Can you write a formula for the amount of money on the n-th day?

Answer: $a_n = 2^{n-1}$

Short explanation:

You multiply by 2 each day

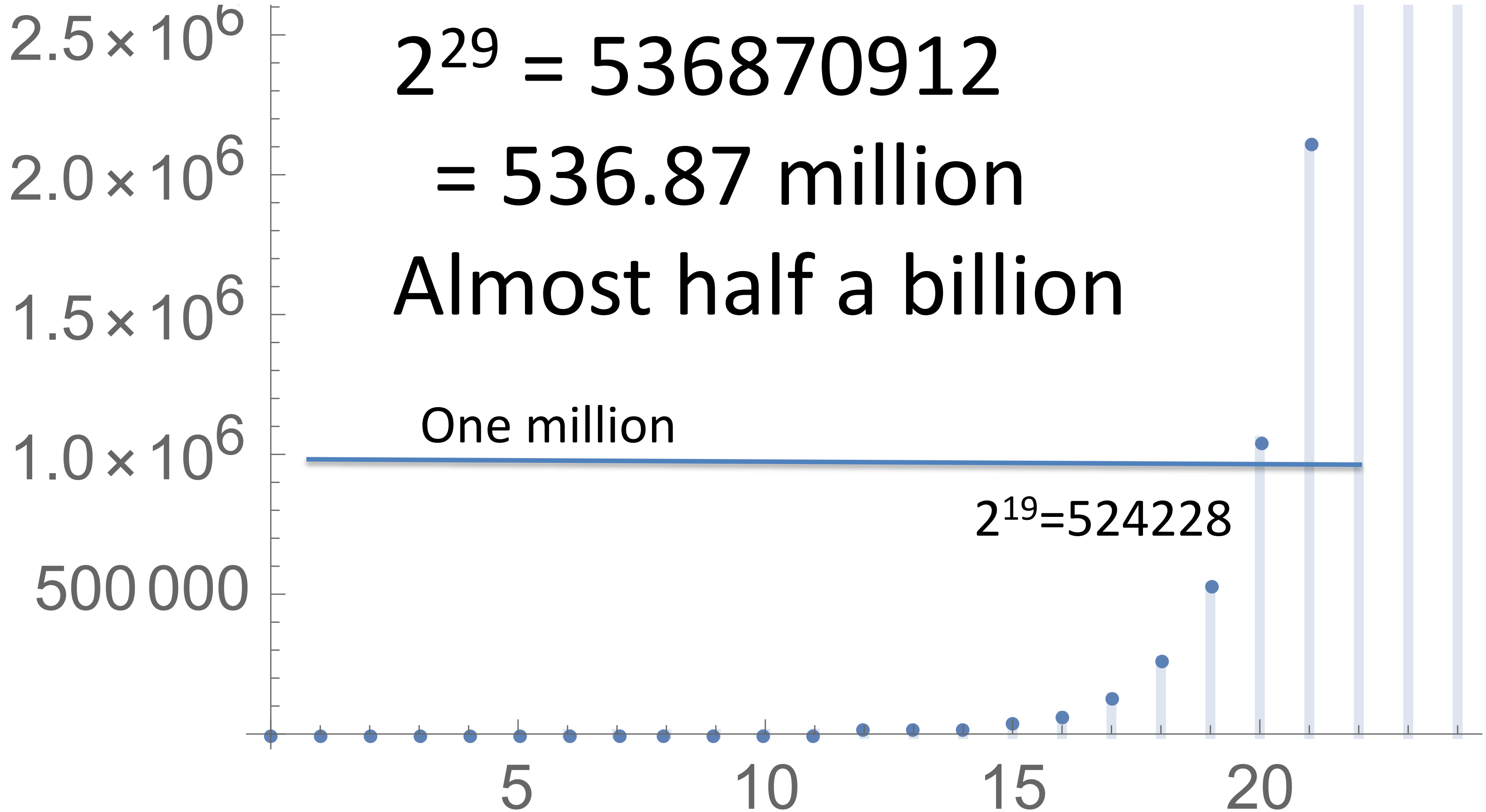
After n days you would have multiplied by 2 exactly n-1 times.

So how much will you have after 30 days?

$2^{29} = 536870912$

$= 536.87$ million

Almost half a billion



Geometric sequences

Sequences that ***grow*** (or decrease)

by *multiplication*

are called Geometric sequences.

General formula for a Geometric sequence

General formula: $a_n = B \times r^n$

(compare with mn+b)

In the case of 2^{n-1} we have $a_n = (1/2) \times 2^n$.

General formula for a Geometric sequence

The formula is also written as

$a_n = ar^{n-1}$

where $a = a_1$ = first term.

So in the sequence 1,2,4,8,…

We have $a_n = 2^{n-1} = 1(2^{n-1})$.

We will use this format for geometric sequence.

Geometric or not?
If geometric, write formula for a_n

- 1,1,1,1,.....,1,......

Ans: Yes. r = 1, a_n =1.

- 0.1, 0.01, 0.001, 0.0001, 0.00001,...

Ans: Yes. r = 0.1 = 1/10 ; a_n = (0.1)^n

- $\frac{1}{8}, \frac{1}{4}, \frac{1}{2}$, 1, ...

Ans: Yes. r = 2. $a_n = 2^{n-1}/8$

- 2, 2^3, 2^5, 2^7, ...

Ans: Yes. $r = 2^2 = 4$. $a_n=2(2^{2(n-1)}) = 2^{2n-1}$

General exponential function

$$f(x) = B \times a^x$$

Here B can be any real number,
a can be any *positive real number*
and x can be *any real number*.

Remember,

$$f(x) = x^m$$

is a *power function*

Examples of exponential functions

Which of the following are exponential functions? If they are, write in the form B x a^x

$f(x) = \frac{3}{2.1^x}$ Yes. B = 3. a= 1/(2.1)

$f(x) = x^{13.24}$ No. Power function.

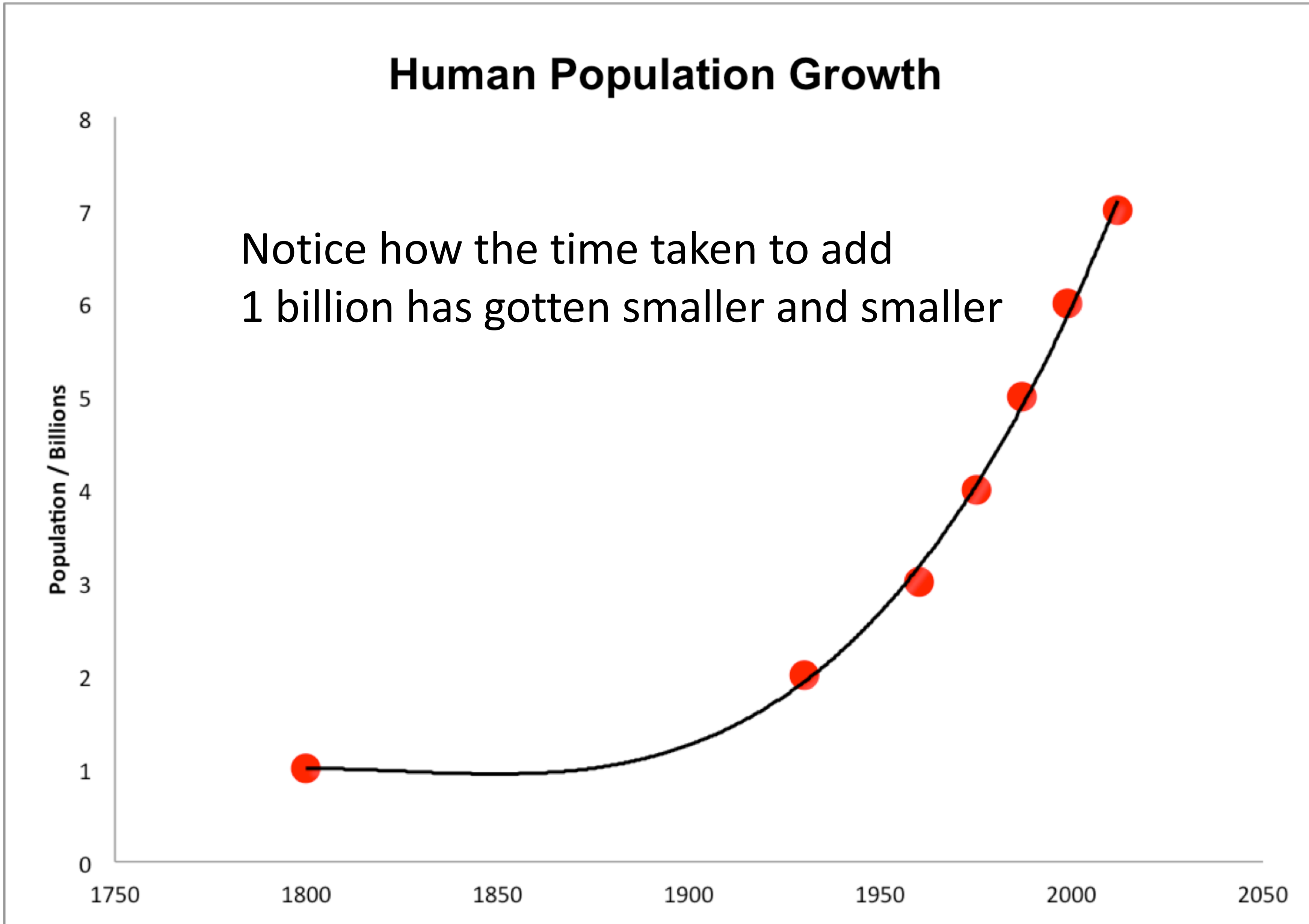
$f(x) = \pi^{x+1}$ Yes. B = π, a= π.

Because $\pi^{(x+1)} = \pi^x$ times π.

Exponential growth

- Exponential function grows so fast that the word “exponentially” itself conveys very fast growth
- Grows faster than any other function we have seen- such as linear, quadratic, cubic, etc.,
- In fact, any exponential function will grow much faster than any polynomial function, *eventually*

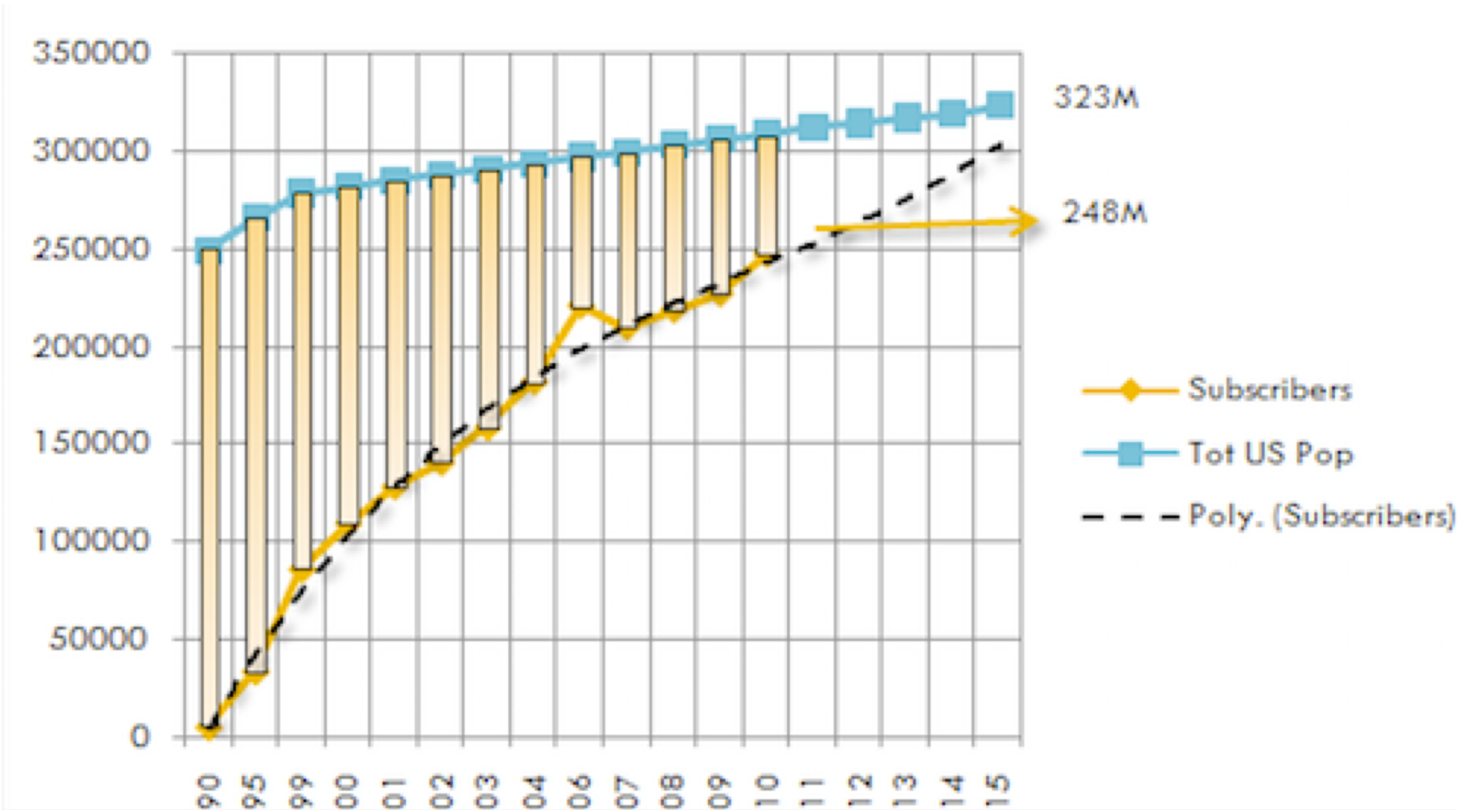
Some things that grow fast



Human population growth and its environmental impacts

- https://www.youtube.com/watch?v=PUwmA3Q0_OE
- https://www.youtube.com/watch?v=B-nEYsyRlYo

Cellphone usage



Why does exponential function grow so fast?

- In a nutshell: It grows by multiplication
- Example: $2^{n+1} = 2^n$ times 2
- Bacteria and other organisms that reproduce by division grow like this

  (provided growth is *uninhibited)*

The Key Point in Exponential Growth

Key point:

Population grows as a factor of itself

Example: Bacteria growing by division

After each division

the previous population is doubled

In other words, multiplied by a factor of 2.

Compound Interest Formula -I

If P is principal,

$A(1)$ = Amount after 1 year

= Principal + Interest

$= P + (r \times P)$

Where r is the rate of interest.

(Different from Common Ratio!)

So $A(1) = P+rP = P(1+r)$

Compound Interest Formula -II

Now amount after 2 years

$A(2) = A(1) + r\,A(1)$

Notice how we are using A(1) instead of P !!

A(2) IS NOT P+2rP as in simple interest !!

So $A(2) = P(1+r) + r(P(1+r)) = P(1+r)(1+r) = P(1+r)^2$

Notice how ***we are multiplying by 1+r*** each time.

In general $A(t) = P(1+r)^t$

Exercise: Compound Interest

P = \$ 1000, r = 5% = 0.05

How much after 1 year?

How much after 2 years?

How much after 3 years?

Exercise: Compound Interest Answer

$P = \$\ 1000,\ r = 5\% = 0.05$

How much after 1 year?

Formula: $A(t) = P(1+r)^t$

$A(1) = 1000(1+0.05)^1 = 1050$

How much after 2 years?

$A(2) = 1050+(.05)(1050) = 1000(1.05)^2 = \$\ 1102.5$

$A(3) = 1102.5 + 0.05(1102.5) = \$\ 1157.63$

Compound Interest Formula Revealed

Let us look at what is really going on:

$A(1) = 1000(1+0.05) = 1050$

$A(2) = 1050+(.05)(1050) = 1050(1.05)$

$= 1000(1.05)^2$

$A(3) = 1102.5 + 0.05(1102.5)$

$= 1102.5(1.05) = 1000(1.05)^3$

So you can see how $A(t) = P(1+r)^t$ comes about.

Compound Interest Formula - III

Compound interest formula:

If P is principal, Amount at time t is given by

- $A(t) = P(1+r)^t$
- $A(t+1) = P(1+r)^{t+1} = P(1+r)^t$ *times* $(1+r)$

  $= A(t)$ times $(1+r) = A(t) + r \times A(t)$

Therefore $A(t+1) - A(t) = r \times A(t)$

That is:

Growth = (r) times (Amount at t)

General compound interest formula

$$A(t) = P(1+(r/n))^{nt}$$

if interest is calculated n times a year

Example

$P = \$\ 1000,\ r = 5\% = 0.05,\ n = 2,\ t = 1$

How much after one year?

General Compound Interest Formula - II

Compound interest formula:

If P is principal, Amount at time t is given by

- $A(t) = P(1+r)^t$
- $A(t+1) = P(1+r)^{t+1} = P(1+r)^t$ *times* $(1+r)$

  $= A(t)$ times $(1+r) = A(t) + r \times A(t)$

Therefore $A(t+1)-A(t) = r \times A(t)$

That is:

Growth = (r) times (Amount at t)

General compound interest formula Answer for example

$P = \$ 1000$, $r = 5\% = 0.05$, $n = 2$, $t = 1$

How much after one year?

$$A(t) = P(1+(r/n))^{nt}$$

$$A(1) = 1000(1+(.05/2))^{2x1}$$

$$=1000(1.025^2)$$

$$= 1000(1.050625)= \$ 1050.63$$

Growth of money by compounding -- Also exponential

In our Example

P = $ 1000, with $r = 5\% = 0.05$,

after 2 compoundings in a year becomes

$=1000(1.025^2) = \$ 1050.63$

So you multiply by 1.025 each time

Some places where you would see compounding

- Finance:
  - Growth of investments, loan payments,…
- Economy: growth of GDP, etc.,
- Growth of populations

Natural exponential function -- 1

What if n is increased in $A(t,n) = P(1+(r/n))^{nt}$?

In other words, you keep compounding more and more times a year …every month, every day, etc.,

Would you make more money?

Natural exponential function -- 2

Say $P = 1$, $r = 1$, $t = 1$ then

Amount after compounding n times in a year

$= A(t,n) = (1+(1/n))^n$

$A(1,1) = (1+1)^1 = 2$,

$A(1,2) = (1+(1/2))^2 = 2.25$,

$A(1,12) = 2.61303529$,

$A(1,365) = 2.714567482$,

$A(1,365 \times 24) = 2.71812669162 \ldots..$

NEVER GOES ABOVE e = 2.71828……

Natural exponential function -- 3

Continuous compounding formula

$$A(t) = Pe^{rt}$$

This is the ***limit*** *of* $A(t) = P(1+(r/n))^{nt}$

as n increases without bound

Back to example: $P = 1000$, $r = 0.05$, $t= 1$

then $A(t) = Pe^{rt} = 1000e^{0.05} = \$\ 1051.27$

The function e^x

- Growth in nature – the growth of trees an animals, the growth of populations, all grow in the same way as continuous compounding, modified by death, competition, loss of resources etc.,
- Any time growth (or decay) happens in nature, the function e^x is involved.

Sum of a geometric sequence

$a + ar + ar^2 + ar^3 + \dots + ar^{n-1}$

$$= a\,\frac{(1-r^n)}{(1-r)}$$

Example sum of geometric sequence

$$1+\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2^{n-1}}$$

$$=\frac{1-\frac{1}{2^n}}{1-\frac{1}{2}}=2-\frac{1}{2^{n-1}}$$

Sum of an infinite series

What would happen as n goes to infinity

In this sum?

$$1+\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2^{n-1}}$$

$$=\frac{1-\frac{1}{2^n}}{1-\frac{1}{2}}=2-\frac{1}{2^{n-1}}$$

Answer to infinite sum

As n goes to infinity, sum will approach 2 !

$$1+\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2^{n-1}} = \frac{1-\frac{1}{2^n}}{1-\frac{1}{2}} = 2 - \frac{1}{2^{n-1}}$$

In the RHS of above sum $\frac{1}{2^{n-1}}$ will get smaller and smaller and ultimately vanish, leaving only 2

Total amount in a retirement fund

- Suppose you invest $1000 each year.
- Say the returns are always 6%.
- How much money would you have
  after 10 years?
- Remember: each of the $1000 gathers interest for different number of years.

Answer for Total amount in a retirement fund

- First \$1000 gives you

  $A(10) = 1000(1.06)^{10}$

- Second \$1000 gives you

  $A(9) = 1000(1.06)^{9}$

And so on. What would be the total?

Answer for Total amount in a retirement fund-II

- We had $A(10) = 1000(1.06)^{10}$,

  $A(9) = 1000(1.06)^9$ and so on.

What would be the total?

The amount deposited in 9th year will give

$A(1) = 1000(1.06)^1$

And the last \$1000 will not give any interest.

So total = $1000 + 1000(1.06) + 1000(1.06)^2 + \ldots$

$\ldots\ldots + 1000(1.06)^9 + 1000(1.06)^{10}$

What is this sum?

Answer for Total amount in a retirement fund-II

Total = $1000 + 1000(1.06) + 1000(1.06)^2 + \ldots$

$\ldots.. + 1000(1.06)^9 + 1000(1.06)^{10}$

This is a sum of a geometric sequence,

With a = 1000, r = 1.06, n = 11.

(n=11 because ar^{10} is the 11^{th} term ; $a_n = ar^{n-1}$).

And using the formula we get

Sum = $a\frac{(1-r^n)}{(1-r)} = 1000\frac{(1-1.06^{11})}{(1-1.06)}$ = \$14,971.64.

Some places where you might need to find sum of a geometric sequence

- Retirement fund (see previous slides)
- Endowment funds
- Monthly payment for loans for cars, homes, etc.,

(In loan payment we are working in reverse:

Instead of depositing fixed amount every period as in previous slides and getting a total at the end, we get a total (loan) upfront and then pay it back in periodic, fixed payments).

Solar power financing (solar loans)

https://www.energysage.com/solar/financing/solar-loans/

Calculation of monthly payment (Just FYI) -- I

- Let monthly payment be m, and total loan M.
- Let i = r/12 be monthly interest rate
- N = number of months (usually 60 months or five years for car loans; 360 months or 30 years for home loans)

Calculation of monthly payment (Just FYI) -- I

As mentioned before, just as amount deposited monthly grows to final amount, we need

Loan given PLUS interest =

the total amount coming

from the monthly payments.

Calculation of monthly payment (Just FYI) -- II

From before,

Loan PLUS interest

= Total amount from the monthly payments.

This gives the following equation

$M(1+i)^n = m(1+i)^{n-1} + m(1+i)^{n-2} + \ldots + m(1+i)^1 + m$

Using geometric sum formula for the sum on the right hand side, with first term m and common ratio 1+i we get

Calculation of monthly payment (Just FYI) -- III

From before,

Loan PLUS interest

= Total amount from the monthly payments.

This gives the following equation

$M(1+i)^n = m(1+i)^{n-1} + m(1+i)^{n-2} + \ldots + m(1+i)^1 + m$

Using geometric sum formula for the sum on the right hand side, with first term m and common ratio 1+i we can get

Monthly payment formula for loans (Just FYI) -- II

We had

$M(1+i)^n = m(1+i)^{n-1} + m(1+i)^{n-2}....+m(1+i)^1 + m$

Using geometric sum formula for the sum on the right hand side, with first term m and common ratio 1+I, we can get

$$M(1+i)^n = m \times \frac{1-(1+i)^n}{1-(1+i)}$$

Dividing all by $(1+i)^n$ and solving for m we get

$$m = \frac{M\,i}{1 - \frac{1}{(1+i)^n}}$$

This is the monthly payment formula

Key properties of exponential growth

- Growth at time t happens as a factor of the amount at time t

- Time to double (or halve) is same no matter what the initial amount.

Finding doubling time

Recall from test: $10000 = Pe^{0.05(13.86)} = Pe^{0.693}$

Also we were given $e^{0.693} = 2$.

So $P = 10000/ e^{0.693} = 10000/2 = 5000$.

0.693 is called

Logarithm of 2 in the base e = $Log_e\ 2 = \ln 2$*.*

Basically it answers the question:

What power of e equals 2?

Finding doubling time -II

So $10000 = 5000\ e^{0.693}$ with t = 13.86 years.

Suppose we want to solve for t in $Pe^{rt} = 2P$

Then cancelling P gives $e^{rt} = 2$

But we know rt = 0.693 for this to happen.

SO DOUBLING TIME IS $t = 0.693/r$

Or (ln 2)/(rate of growth)

Note: ln 2 = 0.693 upto 3 decimal places.

It is actually an infinite, non-repeating decimal.

Basics of Logarithms - I

Logarithms help us answer the question

“What power of a equals b ?”

$a^x = b$ means $x = \log_a b$.

Special cases:

a=10 means $\log_a b = \log b$

a = e means $\log_a b = \ln b$

Nice property:

You can switch from any base to e or 10.

Basics of Logarithms - II

Main properties of logarithms (to any base)

$$\text{Log } A + \text{Log } B = \text{Log } (AB)$$

$$\text{Log } A - \text{Log } B = \text{Log } (A/B)$$

$$\text{Log } A^m = m \text{ Log } A$$

Exercise using logarithm

How long would it take for 2000 dollars

To grow to 6000 dollars at 10% per year

If interest is compounded

1) Annually
2) Continuously

?

Answer to exercise

1) $2000(1+r)^t = 2000(1.1)^t = 6000$ means

$3 = 1.1^t$ means $\ln 3 = \ln (1.1^t) = t \ln(1.1)$

means $t = \ln 3/(\ln 1.1) = 11.53$ years.

2) $2000\ e^{0.1t} = 6000$ means $3 = e^{0.1t}$

means $\ln 3 = \ln (e^{0.1t}) = (0.1)t \times \ln(e) = 0.1t$

means $t = \ln 3/0.1 = 10.99$ years

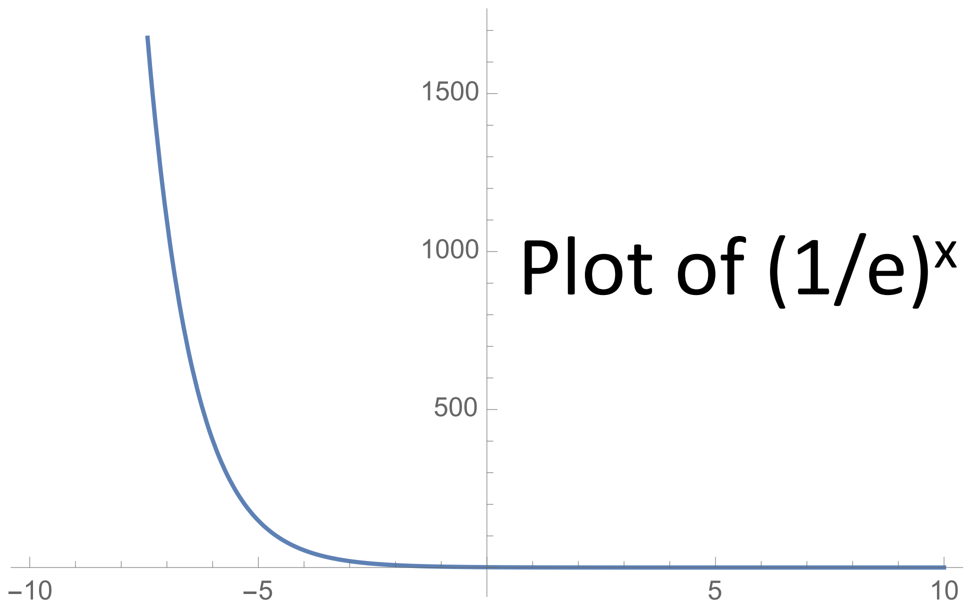
[Here we used that $\ln(e) = 1$ because $e^1 = e$].

Notice the similarity between doubling time and tripling time. You can write similar formula for halving time, time to become one-third, etc.,

Exponential decay

If $a < 1$ in a^x

then we get decay instead of growth



Plot of $(1/e)^x$ or 0.368^x

Half-life

When things decay, instead of doubling time we can calculate **Half-life.**

This is the time it takes for an amount P to become P/2.

Key Point : *No matter what P you start with, it becomes half in the same time.*

Half-life --II

If P becomes P/2, then we get

$$P/2 = Pe^{rt}$$

Dividing by P we get

$$\frac{1}{2} = e^{rt}$$

Now take natural log of both sides.

Half-life --III

We had $\frac{1}{2} = e^{rt}$

Taking natural log of both sides,

$$\ln(1/2) = \ln(e^{rt})$$

means $\ln(1/2) = rt \times \ln(e) = rt.$

Here $\ln(e) = 1$ because $e^1 = e$.

Also ½ $= 2^{-1}$ means $\ln(1/2) = \ln(2^{-1}) = -\ln 2$.

Solving for t, $-\ln 2 / r = t$.

For decay, r will be negative. So t will be positive.

How is half life used to date ancient objects?

Video by Scientific American on

Carbon dating of fossils

(if video doesn't open, copy and paste to browser)

https://www.youtube.com/watch?v=phZeE7Att_s

Exercise from College Algebra

- The bones of a prehistoric man found in the deserts of NM contain about 5% of the original Carbon-14. If half-life of carbon is 5600 years, approximately how long ago did this man die?
- First using $-\ln 2 = r(5600)$ solve for r.
- Get $r = -0.000124$
- Then Solve $0.05 \times P = P \times e^{rt}$

Exercise from College Algebra -- Answer

Qn: The bones of a prehistoric man found in the deserts of NM contain about 5% of the original Carbon-14. If half-life of carbon is 5600 years, approximately how long ago did this man die?

Answer:

First using $-\ln 2 = r(5600)$ (see previous page) solve for r.

Get $r = -0.000124$

Then Solve $0.05 \times P = P \times e^{rt}$

This gives (after canceling P) $0.05 = e^{rt}$

Taking logarithms to "cancel out" the exponential function,

get $\ln(0.05) = rt = -0.000124\ t$ from which we get

$$t = \ln(0.05)/(-0.000124) = 24159.12 \text{ years.}$$

Carbon isotopes and climate change

- Carbon from the CO2 coming from burning fossil fuels has different carbon isotope composition than the carbon in the atmosphere.
- We can find the carbon isotope composition at different times by analyzing tree rings or ice cores from Antarctica.
- Since 1850 the isotope composition has changed dramatically showing that more carbon from fossil fuel burning has entered the atmosphere, thus altering its carbon isotope composition.

For more information go to this page on realclimate.org