4/28/17 Spring 2017, Intro to Modern Algebra II Test III Solutions Sitaraman

EACH PROBLEM 20 POINTS. ANSWER AS MANY AS YOU CAN.

1. Check whether the following polynomials are irreducible. Justify your answer:
   $(a)\ x^3 - x^2 + x - 1 \quad (b)\ x^{100} - 97$

   Solution: (a) $x^3 - x^2 + x - 1 = (x-1)(x^2+1)$. This is reducible.

   (b) $x^{100} - 97$ is irreducible through Eisenstein's criterion. 97 is a prime number that divides all coefficients other than the leading one – all except the constant term are 0, so they are divisible by 97. IThe constant term is not divisible by 97^2.

2. Show that $\mathbb{Q}[x]/(x^3 - x^2 + x - 1)$ is not a field extension of $\mathbb{Q}$.
   What properties of a field does it not have ? Give examples for each.

   Solution: We showed in 1a that $x^3 - x^2 + x - 1$ is reducible. So $(x^3 - x^2 + x - 1)$ is not a maximal ideal, since it is properly contained in the ideal (x^2+1) and $(x-1)$. Since the quotient ring is a field only if the ideal is maximal, we have that $\mathbb{Q}[x]/(x^3 - x^2 + x - 1)$ is not maximal.

   It satisfies all properties of a field except inverse. In fact it has zero divisors:
   Let $J = (x^3 - x^2 + x - 1)$. Then $(x - 1 + J)(x^2 + 1 + J) = x^3 - x^2 + x - 1 + J = J$.
   So neither $x - 1 + J$ nor $x^2 + 1 + J$ have inverses.

3. State whether the following are true or false, with justification for each:

   (a) A finite dimensional extension K of a field F can have a transcendental element, i.e, $\alpha \in K$ that is not algebraic over F.

   (b) $\mathbb{Q}[\pi^2]$ is an algebraic extension over $\mathbb{Q}$.

   Solution: a) A finite dimensional field extension is automatically an algebraic extension, as shown in book. So every element in such a field will be algebraic over F.

   b) We showed in a homework problem that, if a^2 is algebraic over a field F then a is also algebraic over F. So π^2 algebraic means that π would also be algebraic over $\mathbb{Q}$, which is certainly not true.

4. Find the minimal polynomial and hence the degree of the algebraic number $\sqrt[3]{5} + 1$ over $\mathbb{Q}$.

   Solution: Let $\alpha = \sqrt[3]{5} + 1$. Then α satisfies the polynomial $(x-1)^3 = 5 \implies x^3 - 3x^2 + 3x - 6 = 0$. This polynomial is irreducible by Eisenstein's cirterion with $p = 3$ and monic. So it is the irreducible polynomial of α and thus $\mathbb{Q}[\alpha]$ has degree 3 over $\mathbb{Q}$.

5. Give on example for each. Explain how they satisfy the requirements.

   a) An automorphism of $\mathbb{C}$.

   b) An automorphism of $\mathbb{Q}[\sqrt{2}]$.

   Solution:

   The conjugation maps $a + bi \mapsto a - bi$ and $a + b\sqrt{2} \mapsto a - b\sqrt{2}$ are examples of automorphisms for $\mathbb{C}$ and $\mathbb{Q}[\sqrt{2}]$ respectively. Check that they are indeed automorphisms.

6. Prove the following.

   a) (10 points) Find the degree of $\sqrt{2}$ and $\sqrt{3}$ over $\mathbb{F}_{17} = \mathbb{Z}/17\mathbb{Z}$ by finding their irreducible polynomials in $\mathbb{F}_{17}[x]$.

   b) (10 points) Count the number of points in $\mathbb{F}_{17}[\sqrt{2}]$ and $\mathbb{F}_{17}[\sqrt{3}]$.

   Solution:

   a) $2 \equiv 6^2\ mod\ 17$ but $3 \not\equiv a^2\ mod\ 17$ for any $a \in \mathbb{F}_{17}$. So $x - 6$ is the irreducible polynomial for $\sqrt{2}$ and $x^2 - 3$ is the irreducible polynomial for $\sqrt{3}$ over $\mathbb{F}_{17}$.

   b) $\mathbb{F}_{17}[\sqrt{2}]$ is a vector space of dimension 1 over $\mathbb{F}_{17}$ because its degree is 1. So it has just 17 elements. In fact it is simply $\mathbb{F}_{17}$ itself.

   $\mathbb{F}_{17}[\sqrt{3}]$ is a vector space of dimension 2 because its degree is 2. So it will have $17^2 = 289$ elements because you can write all elements as $a + b\sqrt{3}$ and there are 17 choices each for a and b.