4/18/2014 Spring 2014, Linear Algebra Test III Sitaraman

Instructions:
PLEASE PROVIDE STEP BY STEP EXPLANATIONS
ANSWERS WITHOUT EXPLANATION WILL ONLY GET 40 percent
Time Limit 50 minutes ; Total 100 points
Each problem 20 points.
Please read the questions carefully before answering
It is recommended that you first try those problems you are most comfortable with.
Attempt as many as you can; Anything over 100 is extra credit.

1. Say whether each statement is true or false. If true prove your statement or quote the relevant theorem. [It is NOT enough to give just one example]. Otherwise, prove that it is false or provide a counterexample.

   (a) If the $n \times n$ matrix A is obtained from the matrix B by replacing $R1$ (row 1) with $2R1$ then $det(A) = 2det(B)$.

   (b) The nonempty set H such that $H \subseteq V$ is a subspace of V if $a\mathbf{u}+b\mathbf{v} \in H$ whenever $\mathbf{u}, \mathbf{v} \in H$ and $a, b \in \mathbb{R}$.

   Soln:

   (a) TRUE. Basic property of determinants : if a row is multiplied by k then the determinant is multiplied by k also.

   (b)TRUE. We will show that this one property implies all the three properties required for a subset to be a subspace.

   (i) Take $a = 0, b = 0$. Since the set is nonempty, there is atleast one vector $\mathbf{v}$ in H. Then $0\mathbf{v} + 0\mathbf{v} = \mathbf{0}$ is also in H.

   (ii) Take $a = 1, b = 1$. then $\mathbf{u} + \mathbf{v} \in H$ whenever $\mathbf{u}, \mathbf{v}$ are in H.

   (iii) Take $b = 0$. Then $a\mathbf{u} + 0\mathbf{u} = a\mathbf{u} \in H$ whenever $\mathbf{u} \in H$ and a is a scalar.

2. Given that A, B are 3 by 3 matrices with non-zero determinants and $det(A) = 2$, find the following (each 4 points):

   $(a)\ det(A^T)$ $(b)\ det(A^3)$ $(c)\ det(-A)$ $(d)\ det(A^{-1})$ $(e)\ det(BAB^{-1})$

   [For (e) it is enough to know that $det(B)$ is non-zero].

   Solution

   (a) Since $det(A^T) = det(A)$, we get $det(A^T) = 2$.

   (b) $det(A^3) = det(A)^3 = 2^3 = 8$ using the product property for determinants.

   (c) $det(-A) = (-1)^3 det(A) = -det(A)$ because each entry would be multiplied by -1, so when you row expand all terms would have a common factor of $(-1)^3$. (When you multiply a matrix by a number, *every* entry is multiplied by that same number).

In general $det(mA) = m^n det(A)$ is A if an $n \times n$ matrix.

(d) $det(A^{-1}) = 1/det(A) = -1$.

(e) $det(BAB^{-1}) = det(B)det(A)det(B^{-1}) = det(B)det(A)(1/det(B)) = det(A) = 2$.

3. Solve the following system of equations using Cramer's rule:

$$\begin{aligned} x + y + z &= 1 \\ x + 2y - z &= 2 \\ 2x \quad + \quad z &= 0 \end{aligned}$$

Solution:

Note: The solution can be made simpler if you notice that the matrix A equals the matrix $= A_y$.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \\ 2 & 0 & 1 \end{bmatrix} \ ; \ A_x = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix} \ ; \ A_y = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \\ 2 & 0 & 1 \end{bmatrix} \ ; \ A_z = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 2 & 0 & 0 \end{bmatrix}$$

You see that $det(A_x) = |A_x| = 0$ and $det(A_z) = |A_z| = 0$ because both have two columns that are the same. So if $det(A) = |A|$ is not zero then $x = |A_x|/|A| = 0$ and $z = |A_z|/|A| = 0$. You also see that the matrices A and A_y are exactly the same. So we need only to find $|A| = |A_y$.

$$det(A) = |A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \\ 2 & 0 & 1 \end{vmatrix} \xrightarrow{R_3 - R_1 \to R_2} \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 2 & 0 & 1 \end{vmatrix}$$

$$\xrightarrow{R_3 - 2R1 \to R_3} \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & -2 & -1 \end{vmatrix} \xrightarrow{R_3 + 2R2 \to R_3} \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & -5 \end{vmatrix} = 1 \times 1 \times (-5) = -5.$$

So we have $|A| = -5, |A_x| = 0, |A_y| = |A| = -5, |A_z| = 0$.

Thus $x = 0, y = -5/-5 = 1, z = 0$.

Check that these satisfy the equations.

4. Find a basis and hence the dimension of $H \subseteq \mathbb{P}_4$ where H is the subspace of all real polynomials with powers that are even and less than 5 , i.e, polynomials of the form $a_0 + a_2t^2 + a_4t^4$. You must prove that the basis elements span H and that they are linearly independent.

Soln:

A basis is $\{1, t^2, t^4\}$. So the dimension is 3.

Note that these are also polynomials. You can use the isomorphism between $\mathbb{P}_4$ and $\mathbb{R}^5$ to come up with the basis vectors as (1,0,0,0,0), (0,0,1,0,0) and (0,0,0,0,1) but then you have to convert back to polynomials.

Proof that they form a basis:

Let $\mathbf{b}_1 = 1, \mathbf{b}_2 = t^2, \mathbf{b}_3 = t^4$.

We can write any polynomial in H as a linear combination of these basis elements: $a_0 + a_2t^2 + a_4t^4 = a_0\mathbf{b}_1 + a_1\mathbf{b}_2 + a_2\mathbf{b}_3$.

So the three vectors (in this case the vectors in this space are polynomials) span the subspace H. To show that they are linearly independent:

Ssuppose we can find x_1, x_2, x_3 such that $x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + x_3\mathbf{b}_3 = \mathbf{0}$.

Then we get $x_1 + x_2t^2 + x_3t^4 = 0$ for all real number values of t.

(Note that the zero vector in the space $\mathbb{P}_4$ is the zero polnomial which is 0 for all values of t).

But this can only happen if $x_1 = x_2 = x_3 = 0$.

To prove this try t = 0, t = 1, t= 2.

We get $x_1 = 0$ by putting $t = 0$, then $x_2 + x_3 = 0$ and $4x_2 + 16x_3 = 0$ by putting $x_1 = 0, t = 1$ and $x_1 = 0, t = 2$ respectively.

Solving these two equations you get $x_2 = x_3 = 0$ as well.

Since we only have the trivial solution for $x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + x_3\mathbf{b}_3 = \mathbf{0}$, we get that the three polynomials $1, t^2, t^4$ are linearly independent.

[Note: You can also say that since these are already part of the standard basis they are linearly independent. Subset of a linearly independent set is also independent – this is a theorem.]

Since they span the given subspace and are linearly independent, they form a basis for that subspace.

You can also say these are linearly independent because they are a subset of the linearly independent set $\{1, t, t^2, t^3, t^4\}$.

5. Given the transformation $T : \mathbb{P}_2 \to \mathbb{P}_2$ given by $T(f(t)) = f(t + 1)$ where $f(t) = a + bt + ct^2$ is any polynomial in $\mathbb{P}_2$ do the following:

   (a) (6 points) Write $\mathbf{x} = f(t)$ and $\mathbf{y} = f(t + 1)$ in terms of their coordinates in the standard basis $\{1, t, t^2\}$. (Your answer will basically be a vector in $\mathbb{R}^3$ because $\mathbb{P}_2$ is isomorphic to $\mathbb{R}^3$ under the coordinate map).

   (b) (6 points)Write the matrix A corresponding to T so that $A\mathbf{x} = \mathbf{y}$.

   (c) (8 points) Find a basis for $ColA$.

Soln for (a):

$f(t+1) = a+b(t+1)+c(t+1)^2 = a+bt+b+ct^2+2ct+c = (a+b+c)+(b+2c)t+ct^2.$

$$\mathbf{x} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}; \quad \mathbf{y} = \begin{bmatrix} a+b+c \\ b+2c \\ c \end{bmatrix}$$

Solution for (b):

The matrix is $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$.

We see that $A\mathbf{x} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a+b+c \\ b+2c \\ c \end{bmatrix} = \mathbf{y}$.

Solution for (c):

The matrix A is already in echelon form with pivot in each row and column. So the rank is 3 and the column space $ColA$ has dimension 3 and a bsisi is given by *all* the columns of A. i.e, the basis is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}$$

6. (Optional) (20 points) Find the eigenvalues and the eigenvectors corresponding to each eigenvalue for the matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$.

Soln:

The eigenvalues are just the numbers on the diagonal namely 1 and 2.

If you used the equation $det(A - \lambda I) = 0$ you would get $(1-\lambda)(2-\lambda) = 0$ from which you get the same values for λ.

To find eigenspace for $\lambda = 1$ find null space of $A - 1I = \begin{bmatrix} 1-1 & 1 \\ 0 & 2-1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$

The null space of this matrix would be given by $x = x, y = 0$.

So the eigenspace is the set of vectors of the form $(x, 0)$ where x is any real number.

It can be written as a span of $(1, 0)$ which is an eigenvector for 1.

Note that **all vectors** of the form $(x, 0)$ are eigenvectors for $\lambda = 1$.

To find eigenspace for $\lambda = 2$ find null space of $A - 2I = \begin{bmatrix} 1-2 & 1 \\ 0 & 2-2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$

The null space of this matrix would be given by $x = y, y = y$.

So the eigenspace is the set of vectors of the form (y, y) where y is any real number.

(You could have used x as a free variable too).

It can be written as a span of $(1, 1)$ which is an eigenvector for 2.

Check your answer by computing $A\mathbf{x} = \lambda\mathbf{x}$ for each eigenvalue λ and each eigenvector $\mathbf{x}$.