2/12/2014 Spring 2014, Linear Algebra Test I Sitaraman

Instructions:
PLEASE PROVIDE STEP BY STEP EXPLANATIONS
ANSWERS WITHOUT EXPLANATION WILL ONLY GET 40 percent
Time Limit 50 minutes ; Total 100 points
Each problem 20 points.
Please read the questions carefully before answering
It is recommended that you first try those problems you are most comfortable with.
Attempt as many as you can; Anything over 100 is extra credit.

1. Say whether each statement is true or false. If true prove your statement or quote the relevant theorem. [It is NOT enough to give just one example]. Otherwise, prove that it is false or provide a counterexample.

   (a)
$$\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} \text{ is in the span of } \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

   (b) Replacing R_1 by $\dfrac{R_1 - R_2}{3}$ is a permissible row operation while row reducing an augmented matrix.

   Soln:

   (a) FALSE. Suppose we have the following vector equation:

$$\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} = x \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

   Then comparing first and second components, we get $x = 0, y = 1$ but then comparing third component we must have $0(-1) + 1(-2) = -1$ which is not true.

   You can also do this by putting these three vectors as columns in the corresponding augmented matrix and showing that after row reduction it is inconsistent.

   (b)TRUE. This is really $\frac{1}{3}R_1 - \frac{1}{3}R_2$. it only involves scalar multiplication and addition or subtraction of rows.

2. Reduce the following augmented matrix to a reduced echelon matrix and find the general solution to the corresponding system of equations:

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

Soln: To get 0 in R_{31} (row 3, first element) we replace R3 by R3-R1, getting:

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & -1 & 1 \end{bmatrix}$$

To get zero at R_{32} (row 3, second element) replace R2 by R2-R1, resulting in:

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -2 & 0 \end{bmatrix}$$

Dividing row 3 by -2 we get

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

This is the echelon matrix.

To get reduced echelon form, we need to get 0's above the diagonal, in particular at R_{23} and R_{13}.

Replacing R2 by R2-R3 and R1 by R1-R3 gives

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

This is the reduced echelon matrix. We see that the solution is $x_1 = 0, x_2 = 1, x_3 = 0$.

Check that this satisfies original equation.

3. Write the following system of equations as

   (a) a vector equation

   (b) a matrix equation.

$$\begin{aligned} x_1 &= x_4 \quad (1) \\ x_2 - x_3 &= 1 \quad (2) \\ x_1 + x_3 + x_4 &= 2 \quad (3) \end{aligned}$$

No need to solve it!

Soln: First write equation (1) as $x_1 - x_4 = 0$.

Be careful with the coefficients! the variables are not aligned properly in the same column.

After rearranging, equations look like this:

$$\begin{aligned} x_1 \quad & \quad - x_4 &= 0 \qquad (1) \\ x_2 & \quad - x_3 &= 1 \qquad (2) \\ x_1 \quad & + x_3 + x_4 &= 2 \qquad (3) \end{aligned}$$

The matrix equation is is

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & -0 \\ 1 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

The vector equation is

$$x_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

4. Check whether the following vectors are linearly independent: (each is a separate problem)

$$(a) \qquad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

$$(b)\ \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Soln:

(a) independent: they are not multiples of each other. Since there are only two vectors, that is enough to check.

You can also do this by reducing the matrix below and showing that only solution is the trivial solution (0,0):

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 5 & 0 \end{bmatrix}$$

(b) dependent because we have three vectors in two dimensions (theorem 8).

You can also do this by showing that the reduced echelon matrix for the augmented matrix below upon row reducing will have nontrivial solutions:

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 5 & 1 & 0 \end{bmatrix}$$

5. In a certain country let P, M, R represent the number of people who are poor, middle-class and rich respectively. Suppose after 20 years, 19 percent of the poor people and 30 percent of the rich people become middle-class ; 5 percent of the rich and 15 percent of the middle-class become poor ; and 1 percent of the poor and 10 percent of the middle class become rich.

   (a) (6 points) Write an exchange table for this phenomenon.

   (b) (8 points) Suppose the total number of each category, namely P, M and R remain the same even though people are migrating from one to the other. Express this in the form of three equations in P, M and R.

   (c) (6 points) Write the augmented matrix for the system equations from (b). DON'T solve it.

   Soln: (a) The exchange table: (the numbers indicate percentage moving from one group to another.

Poor	*Middle*	*Rich*	$\rightarrow$
0.8	0.15	0.05	*Poor*
0.19	0.75	0.30	*Middle*
0.01	0.10	0.65	*Rich*

   (b) The equations when the numbers remain the same:

$$\begin{aligned} 0.8P + 0.15M + 0.05R &= P \quad (1) \\ 0.19P + 0.75M + 0.30R &= M \quad (2) \\ 0.01P + 0.1M + 0.65R &= R \quad (3) \end{aligned}$$

   Simplifying so that all variables are on one side we get

$$\begin{aligned} -0.2P + 0.15M + 0.05R &= 0 \quad (1) \\ 0.19P - 0.25M + 0.30R &= 0 \quad (2) \\ 0.01P + 0.1M - 0.35R &= 0 \quad (3) \end{aligned}$$

   The augmented matrix is

$$\begin{bmatrix} -0.2 & 0.15 & 0.05 & 0 \\ 0.19 & -0.25 & 0.30 & 0 \\ 0.01 & 0.1 & -0.35 & 0 \end{bmatrix}.$$

6. (Challenge, 20 points) Given that a cubic polynomial $f(x) = ax^3 + bx^2 + cx + d$ satisfies $f(1) = 0, f'(1) = 2, f'(2) = 0, f"(1) = 0$ find the coefficients a, b, c, d and hence the polynomial.

Soln:

We have $f(1) = a+b+c+d$, $f'(1) = 3a+2b+c$, $f'(2) = 12a+4b+c$, $f''(1) = 6a+2b$.

Using given information we get

$$\begin{aligned} a+b+c+d = &\ 0 \quad (1)\\ 3a+2b+c = &\ 2 \quad (2)\\ 12a+4b+c = &\ 0 \quad (3)\\ 6a+2b = &\ 0 \quad (4) \end{aligned}$$

$$\text{(3)-(2) gives } 9a + 2b = -2 \quad (5)$$

$(5) - (4)$ gives $3a = -2 \implies a = -2/3$.

Putting $a = -2/3$ in (4) gives $-4 + 2b = 0 \implies b = 2$.

Plugging in $b = 2, a = -2/3$ in (2) we get $-2 + 4 + c = 2 \implies c = 0$.

Now from (1) we get $\frac{-2}{3} + 2 + 0 + d = 0 \implies d = \frac{-4}{3}$.

So the desired polynomial is $\frac{-2}{3}x^3 + 2x^2 + \frac{-4}{3}$.