1.(10 points) Find the co-ordinates of $2x+3$ in the basis $\{1+x, 1-x\}$ of $\mathbf{P}_2$.

Soln: The required co-ordinates are given by $P^{-1}\mathbf{v}$ where $P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ is the matrix whose columns are the vectors corresponding to $1+x$ and $1-x$, and $\mathbf{v} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ is the vector corresponding to $2x+3$. The answer is $\begin{bmatrix} 5/2 \\ 1/2 \end{bmatrix}$. Check that $5/2(1+x) + 1/2(1-x) = 2x+3$.

2. (15 points) Find the dimension of Nul(A) where A is the matrix given below. Then find the rank of A, without calculating the dimension of Col(A) :

$$\begin{bmatrix} 1 & 0 & -2 \\ -1 & 2 & 2 \\ 0 & 3 & 0 \end{bmatrix}$$

Soln: Row reducing the augmented matrix for the homogenous equation we get

$$\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

from which we get that the parametric solution is $x - 2z = 0$ which gives a basis for the null space of A as $\{\mathbf{v}\}$ where $\mathbf{v} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$. Check that $A\mathbf{v} = \mathbf{0}$. Thus null space has dimension 1 and rank = 3-1 = 2.

3. (15 points) Find a basis for the eigenspace of the eigenvalue $\lambda = -2$ of the matrix

$$\begin{bmatrix} 1 & 0 & -1 \\ 1 & -3 & 0 \\ 4 & -13 & 1 \end{bmatrix}$$

Soln: If the given matrix is A we need to find a basis for the null space of $A-\lambda I$. Row reducing the augmented matrix for the homogenous equation $(A-\lambda I)\mathbf{x}=\mathbf{0}$, we get

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 3 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

from which we get a basis for the nullspace (hence the eigenspace of -2) as $\{\mathbf{v}\}$ where $\mathbf{v}=\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$. Check that $\mathbf{v}$ is an eigenvector for -2.

4. (15 points) State whether the following are true of false. If true, prove it, and if false disprove it or provide counterexample. (Usually it is not enough to provide example if you want to prove something).
(a) For an $m \times n$ matrix, its rank is smaller than or equal to the smaller of m and n.
(b) Given any set of m vectors in a vector space V of dimension n, we can produce a basis of V as long as $m > n$.
(c) If $\mathbf{v}$ is an eigenvector of A, then $A\mathbf{v} = k\mathbf{v}$ for all real numbers k.

Soln: (a) is true because the number of pivot entries will be smaller than or equal to the minimum of m and n. (b) is false. Counterexample: Consider a set of m vectors where all the vectors are multiples of a single vector, hence they are all linearly dependent. It is not possible to produce a basis from this. (c) False because eigenvector corresponds to a particular eigenvalue, not any real number.

5. (15 points) Find the eigenvectors and diagonalize $\begin{bmatrix} -3 & 4 \\ -6 & 7 \end{bmatrix}$.

Soln: The characteristic polynomial is obtained by taking the determinant of $A-\lambda I = \begin{bmatrix} -3-\lambda & 4 \\ -6 & 7-\lambda \end{bmatrix}$. The roots of this polynomial give the eigenvalues and they are 1,3. Finding the null space of $A-1I$ and $A-3I$ we get the eigenvectors as $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ respectively for 1 and 3.

Then the diagonal matrix similar to A is given by $A = P^{-1}AP = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$ where $P = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$.

6.(15 points) Write $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ as a sum of two vectors, one along $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and the other along $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ using orthogonal projection of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

The solution is given by $\mathbf{y} = \hat{\mathbf{y}} + (\mathbf{y} - \hat{\mathbf{y}})$ where $\hat{\mathbf{y}}$ is the orthogonal projection of the given vector $\mathbf{y} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ along $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. The two components in the sum are $\begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix}$ and $\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$ respectively.

7.(15 points) Find an orthonormal basis for the subspace of $\mathbf{R}^3$ spanned by $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Soln: The answer is obtained using the Gram-Schmidt process and in fact it is very similar to the answer for 6 except that we take the unit vectors $\frac{\mathbf{x}_1}{||\mathbf{x}_1||}$ and $\frac{\mathbf{x}_2 - \hat{\mathbf{x}}_2}{||\mathbf{x}_2 - \hat{\mathbf{x}}_2||}$ where $\hat{\mathbf{x}}_2$ is the orthogonal projection of $\mathbf{x}_2$ along $\mathbf{x}_1$.

8. [Challenge, 20 points] Prove that the orthogonal complement of a subspace of a vector space is also a subspace.

Soln: If $\mathbf{u}, \mathbf{v}$ are in the orthogonal complement W_1 of W then $\mathbf{u} \bullet \mathbf{w} = 0$ and $\mathbf{v} \bullet \mathbf{w} = 0$ for every $\mathbf{w}$ in W. Then $(\mathbf{u} + \mathbf{v}) \bullet \mathbf{w} = 0$ as well and so $\mathbf{u} + \mathbf{v}$ is also in W_1. Also $c\mathbf{u} \bullet \mathbf{w} = 0$ for any scalar c so $c\mathbf{u}$ is also in W_1. Thus W_1 is also a subspace of the vector space.