5/4/2011 Spring 2011, Linear Algebra Final Exam Sitaraman

Instructions:
PLEASE PROVIDE STEP BY STEP EXPLANATIONS
ANSWERS WITHOUT EXPLANATION WILL ONLY GET 40 percent
Time Limit 120 minutes ; Total 200 points
Please read the questions carefully before answering
It is recommended that you first try those problems you are most comfortable with.

Problem 1 is mandatory. Answer any 8 of remaining 10 which carry 20 points each.

1. (40 points) Say whether each statement is true or false. If true prove your statement or quote the relevant theorem. [It is NOT enough to give just one example]. Otherwise, prove that it is false or provide a counterexample.
(a) A system of linear equations with more equations than variables will always have infinitely many solutions.
(b) For any two matrices A and B we have $AB = BA$.
(c) T is any linear transformation from $\mathbf{R}^n$ to $\mathbf{R}^n$. If there is a non-zero vector $\mathbf{v}$ such that $T(\mathbf{v}) = \mathbf{0}$ then T cannot be one-one.
(d) If A is a diagonalizable matrix, [i.e, $A = PDP^{-1}$ with D being diagonal] then A^k is also diagonalizable and $A^k = PD^kP^{-1}$.

Soln:
(a) FALSE. It is the other way. If there are more variables than equations then there will be free variables and you can get infinitely many solutions. If there are more equations than variables you can still get a unique solution or no solutions at all.
(b) FALSE. Easy to make two matrices which don't commute.
(c) TRUE. We have $T(\mathbf{0}) = \mathbf{0}$ for any transformation, already. So we have both $\mathbf{v}$ and $\mathbf{0}$ going to $\mathbf{0}$.
(d) TRUE. Easy to see if you multiply it out.

2. Solve the following system of equations:

$$\begin{aligned} 2x + 3y + 2z &= 4 \\ x \qquad - z &= 1 \\ 3y + z &= 0 \end{aligned}$$

Soln: $x = 5/3,\ y = -2/9,\ z = 2/3$.

3. A map T is defined by $T(x, y, z) = (x+y, xy, 0)$. Show that it is not a linear transformation by checking if T satisfies the definition of a linear transformation. [It is not enough to show that there isn't a matrix A such that $T(\mathbf{v}) = A\mathbf{v}$].

We check the condition $T(k\mathbf{v}) = kT(\mathbf{v})$.
We have $T(kx, ky, kz) = (kx + ky, k^2xy, 0)$.
But $kT(x, y, z) = (k(x + y), kxy, 0)$.
The second components don't match.

4. Solve by using Cramer's rule :

$$\begin{array}{cccccc} x & -y & & = & 3 \\ 2x & + & +3z & = & 4 \\ & y & +z & = & 2 \end{array}$$

Solution:

$$x = \frac{-11}{-1} = 11,\ y = \frac{-8}{-1} = 8,\ z = \frac{6}{-1} = -6.$$

5. Find the inverse of the matrix $\begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$ using row operations. Indicate each operation.

Soln: $\begin{bmatrix} -1 & 1 & -1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$

6. Show that the polynomials $1 + t - t^2, 1 + t, t^2 - 1$ form a basis for $\mathbf{P}_2$, the space of polynomials of degree *atmost* 2. [in other words, every such polynomial can be written as a linear combination of these three].

Soln:

Writing the polynomials as vectors in $\mathbf{R}^3$ we get (1,1,-1), (1,1,0) and (-1,0,1). Note that these are the columns of the matrix in problem 6. Since that matrix has an inverse it must have linearly independent column vectors. Thus the corresponding polynomials must form a basis for $\mathbf{P}_2$ because $\mathbf{P}_2$ has dimension 3 (it is actually isomorphic to $\mathbf{R}^3$).

7. Show that the space of **all** polynomials with real number coefficients is a vector space. You don't need to prove this completely but show that it has a zero vector and that it is closed under addition and scalar multiplication (scalars being the real numbers). [Bonus 10 points: What do you think of the dimension of this vector space?]

Soln:

Adding two polynomials or multiplying a polynomial by a scalar results in a polynomial. This space will be infinite dimensional.

8. Show that the following vectors span $\mathbf{R}^2$: $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

Soln: Actually any two of them will be a basis for $\mathbf{R}^2$ because this is a linearly independent set, and $\mathbf{R}^2$ has dimension only 2!

9. Find the eigenvectors and eigenvalues of $\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix}$.

Soln: Eigenvalues are 0, 1 and -1.

Eigenvectors are $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1/2 \\ 1/2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix}$

(or any multiples of these).

10. A matrix A has eigenvalues 1,2 and corresponding eigenvectors $\mathbf{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$. Show that it is diagonalizable. Find the matrix A as well as the corresponding diagonal matrix. Show that $A\mathbf{u} = \mathbf{u}$ and $A\mathbf{v} = 2\mathbf{v}$.

Soln: $D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$. The matrix is

$$A = PDP^{-1} = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}^{-1} = \begin{bmatrix} -8 & 6 \\ -15 & 11 \end{bmatrix}$$

11. Show that the following is an orthonormal basis for $\mathbf{R}^3$:

$$\begin{bmatrix} 1/\sqrt{2} \\ -1\sqrt{2} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}.$$