4/1/2011 Spring 2011, Calculus III Test III Sitaraman

Instructions:
PLEASE PROVIDE STEP BY STEP EXPLANATIONS
ANSWERS WITHOUT EXPLANATION WILL ONLY GET 40 percent
Time Limit 50 minutes ; Total 100 points
Please read the questions carefully before answering
It is recommended that you try those problems you are most comfortable with, first.
Attempt as many as you can; Anything over 100 is extra credit.

1. (30 points) Say whether each statement is true or false. In each case, explain why it is true/false or give counterexample. If true it is NOT enough to give just one example. You may quote theorems from the book in support of your argument.

(a) If the limit of $f(x, y)$ along the $x-$ axis as $(x, y) \to (0, 0)$ equals L and the limit of $f(x, y)$ along the $y-$axis equals L as $(x, y) \to (0, 0)$ then the limit as $(x, y) \to (0, 0)$ exists and equals L.

(b) The function $f(x, y) = \sqrt{x^2 + y^2}$ is continuous and differentiable at (0,0).

(c)The gradient ∇f of a function $f(x, y)$ gives the slope of the tangent of the curve $f(x, y) = c$ at any point (a, b) on this level curve wherever $\nabla f \neq \mathbf{0}$.

Soln:

1a) FALSE. Here is an example: Let $f(x, y) = -\frac{xy}{x^2+y^2}$. As shown in example 1 of 13.2, this goes to 0 along $x-$axis and $y-$axis but goes to -1/2 along the line $x = y$. So it has no limit.

1b) FALSE. It is continuous at (0,0) but not differentiable. $f_x = \frac{x}{\sqrt{x^2+y^2}}$ does not go to a limit at (0,0), so it does not exist. For a function to be differentiable, f_x has to exist at (0,0). [Proof that limit does not exist similar to the one in (a)].

1c) FALSE. As discussed in class (and shown in book), the gradient is perpendicular to the tangent vector to the level curve at every point on the level curve.

2 (20 points) Complete the following statement (Theorem on differentiablility): A function $f(x, y)$ is differentiable at (x_0, y_0, z_0) if the first partial derivatives f_x, f_y, f_z......

Using this theorem prove that $f(x, y, z) = xysinz$ is differentiable everywhere. Find the derivative along the unit vector $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$ at (1,1,0).

Theorem 13.4.4: If all first order partial derivatives exist and are continuous at a point, then f is differentiable at that point.

$f_x = ysinz$; $f_y = xsinz$; $f_z = xycosz$.

All of these functions are well-defined and continuous everywhere.

So the function is differentiable everywhere.

We have $\nabla f =< f_x, f_y, f_z >=< 0, 0, 1 >$ at (1,1,0).

If $\mathbf{u} = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1)$, then $D_{\mathbf{u}} f(1, 1, 0) = \nabla f \cdot < \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 >$

$=< 0, 0, 1 > \cdot < \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 >= 0.$

3. (10 points) Show that $z = e^{-t} sin(x/c)$ satisfies the famous **heat equation** from Physics [Here $c > 0$ is a fixed constant]:

$$\frac{\partial z}{\partial t} = c^2 \frac{\partial^2 z}{\partial x^2}$$

Soln:

Let $z = f(x,t)$. Then $f_t = -e^{-t}sin(x/c)$. $f_x = e^{-t}cos(x/c)(1/c)$

$\Rightarrow f_{xx} = (1/c)e^{-t}[-sin(x/c)(1/c)] = \frac{-e^{-t}sin(x/c)}{c^2}$.

Plugging in the expression for f_t in this we get $f_x = f_t/c^2$ and thus $f_t = c^2 f_{xx}$ as required.

4 (20 points) Find parametric equations for the tangent line to the curve of intersection of the cylinders $x^2 + z^2 = 25$ and $y^2 + z^2 = 25$ at the point (3,-3,4).

Soln :

Note: The curve of intersection is given by $x^2 = y^2$ which is a $pairoflines$! But clearly the curve of intersection is NOT a pair of lines. You need to use both equations to represent the cuve of intersection.

The tangent line to the curve of intersection lies on the tangent planes to both of these surfaces at this point. So it is perpendicular to the normals to both of their tangent planes. Now the normals are given by the gradients. A vector perpendicular to both of them will be given by their cross product.

Let $F = x^2 + z^2$ and $G = y^2 + z^2$. Then a vector parallel to the tangent line of curve of intersection is $\mathbf{v} = \nabla F \times \nabla G =< 2x, 0, 2z > \times < 0, 2y, 2z >$. Now at (3,-3,4), $\nabla F \times \nabla G =< 6, 0, 8 > \times < 0, -6, 8 >$.

The cross product is given by

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6 & 0 & 8 \\ 0 & -6 & 8 \end{vmatrix}$$

$=< 0 - (-48), -(48 - 0), (-36 - 0) >= 48\mathbf{i} - 48\mathbf{j} - 36\mathbf{k}$.

Then the equation for the tangent line at (3,-3,4) is given by

$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v} =< 3, -3, 4 > +t < 48, -48, -36 >$.

The parametric equations are: $x = 3 + 48t, y = -3 - 48t, z = 4 - 36t$.

5. (20 points) Find three positive numbers x, y, z whose sum is 27 and whose sum of squares $x^2 + y^2 + z^2$ is the smallest.

Soln:

Method 1: using Lagrange multipliers:

Letting $f = x^2 + y^2 + z^2$ and $g = x + y + z$ we need $< f_x, f_y, f_z >= \lambda < g_x, g_y, g_z >$.

This gives $< 2x, 2y, 2z >= \lambda < 1, 1, 1 >$. This equation can only be possible if $x = y = z$ in which case they all equal 9 because they have to add up to 27.

We can take the box $0 \leq x \leq 27, 0 \leq y \leq 27, 0 \leq z \leq 27$ to look for values of extrema because each of the numbers can be atmost 27 and they are greater than or equal to 0. Reason for taking this box is that it is a closed and bounded set.

Now, from the equations above we see that there is only one extremum in it, and that is at (9,9,9). Comparing with just one other point (0,0,27) we see that $x^2 + y^2 = z^2$ is $27^2 = 729$ at the first point and $9^2 + 9^2 + 9^2 = 243$ at the other point. So (9,9,9) must be an absolute minimum.

Method 2: using absolute extrema theorems:

We need to find the *absolute* minimum of $x^2+y^2+z^2$ given that $x+y+z=27$. Solving for z we get $z=27-x-y$. Plugging this into the first equation we get $f(x,y)=x^2+y^2+(27-x-y)^2=x^2+y^2+729+x^2+y^2+2xy-54x-54y=2x^2+2y^2+2xy-54x-54y+729$.

First we find the critical points:

$f_x=0$ and $f_y=0$ gives $4x+2y-54=0, 4y+2x-54=0$. Solving these two equations we get $y=9$. Plugging this back into the equation we get $x=9, y=9, z=9$. Also $x^2+y^2+z^2=243$.

Now we need to show that 243 is the *absolute* minimum.

First note that x,y,z are positive and also $x+y\leq 27$ because $x+y+z\leq 27$ already and $z>0$. So we need to show that the maximum of $f(x,y)$ occurs within the triangle bounded by $x-$axis, $y-$axis and $x+y=27$.

When $x=0$ we have $f(x,y)=2y^2-54y+729$. This is a one variable function and can be minimized using one variable derivatives. We have $f_y=4y-54=0\Rightarrow y=13.5$. This gives $z=13.5$ and so $x^2+y^2+z^2=364.5$ This is also bigger than 243.

Similarly when $y=0$ we would get $x^2+y^2+z^2=364.5$ as the value at the critical point $(13.5,0,13.5)$ on $y=0$. [The equations are symmetrical in x and y].

When $x+y=27$ we have $z=0$ and get $f(x,y)=x^2+y^2$ with $x+y=27$. Plugging $x=27-y$ we get $f(x,y)=(27-y)^2+y^2=729-54y+2y^2$ which is same as before, thus minimum is 364.5 again.

The only boundary points left are the vertices: (0,0),(0,27) and (27,0).

But at all these points either all three numbers are zero (which is invalid) or one of them is 27. When one of them is 27 the other two are 0 and $x^2+y^2+z^2=27^2=729$ which is bigger than 243.

Thus at all the critical points (both inside the triangle and on its boundary) the minimum is 243.

6. (Challenge, 20 points) [From Apostol's "Calculus"] Show that if $f(x,y)$ is defined by $f(x,y)=xy\frac{x^2-y^2}{x^2+y^2}$ for $(x,y)\neq(0,0)$ and at (0,0) it is defined as $f(0,0)=0$ then $f_{xy}=-1$ but $f_{yx}=1$ at (0,0).

Soln:

Using basic definition we get

$$f_{xy}=(f_x)_y=\underset{y\to 0}{Lim}\frac{f_x(0,y)-f_x(0,0)}{y}$$

But we have

$$f_x(0,0)=\underset{x\to 0}{Lim}\frac{f(x,0)-f(0,0)}{x}=0$$

When $(x,y)\neq(0,0)$ we can differentiate the function directly. So at $(0,y)$ with $y\neq 0$ we get using quotient rule

$$f_x(0,y)=xy\frac{x^2-y^2}{x^2+y^2}=\left(\frac{x^3y-xy^3}{x^2+y^2}\right)_x(0,y)$$

$$=\left.\frac{(3x^2y-y^3)(x^2+y^2)-(x^3y-xy^3)(2x)}{(x^2+y^2)^2}\right|_{(0,y)}=(-y^5)/y^4=-y$$

Thus

$$f_{xy} = (f_x)_y = \underset{y \to 0}{Lim} \frac{f_x(0,y) - f_x(0,0)}{y} = \underset{y \to 0}{Lim} \frac{-y - 0}{y} = -1$$

Similarly we can show that $f_{yx}(0,0) = 1$.

Thus the mixed partial derivatives are different here. It can be shown that they are not continuous. When the second derivatives are continuous then the order won't matter.