2/25/2011 Spring 2011, Calculus III Test II Sitaraman

Instructions:
PLEASE PROVIDE STEP BY STEP EXPLANATIONS
ANSWERS WITHOUT EXPLANATION WILL ONLY GET 40 percent
Time Limit 50 minutes ; Total 100 points
Please read the questions carefully before answering
It is recommended that you try those problems you are most comfortable with, first.
Attempt as many as you can; Anything over 100 is extra credit.

1. (30 points) Say whether each statement is true or false. In each case, explain why it is true/false or give counterexample. If true it is NOT enough to give just one example. You may quote theorems from the book in support of your argument.

(a)If $||\mathbf{r}(t)|| = 1$ for any particular value $t = t_0$ then $||\mathbf{r}'(t_0)|| = 1$ also.

(b) For any two vector valued functions $\mathbf{u}(t)$ and $\mathbf{v}(t)$ we have
$(\mathbf{u}(t) \cdot \mathbf{v}(t))' = \mathbf{u}'(t) \cdot \mathbf{v}'(t)$.

(c) A particle is traveling along a curve given by position vector $\mathbf{r}(t)$ and unit tangent vector $\mathbf{T}$ at any point t. Then the velocity vector at t is $\mathbf{v} = ||\frac{d\mathbf{r}}{dt}||\mathbf{T}$.

Soln:

1a) FALSE. Here is an example: Let $\mathbf{r}(t) = t^2\mathbf{i}$.
Then at $t = 1$ we have $||\mathbf{r}(1)|| = ||\mathbf{i}|| = 1$. But $||\mathbf{r}'(1)|| = ||(2t\mathbf{i})_{t=1}|| = ||2\mathbf{i}|| = 2$.

1b) FALSE. $(\mathbf{u}(t) \cdot \mathbf{v}(t))' = \mathbf{u}(t) \cdot \mathbf{v}'(t) + \mathbf{u}'(t) \cdot \mathbf{v}(t)$.

1c) TRUE. As discussed in class (and shown in book), $||\frac{dr}{dt}|| = ||\frac{ds}{dt}||$ is the speed. Since the particle is traveling at any given time in the direction of the tangent vector, velocity is given by speed times unit tangent vector which is what we have here.

2(a). (15 points) Express the equation $36z^2 = r^2(4cos^2\theta + 9sin^2\theta)$ in rectangular and spherical co-ordinates.

2(b). (10 points) Show that the curve with parametric vector equation
$\mathbf{r}(t) = 3tcos\alpha\mathbf{i} + 2tsin\alpha\mathbf{j} + t\mathbf{k}$, where α is fixed, lies entirely on this surface.

Soln for (a):

We have $x = r\cos\theta$ and $y = rsin\theta$. Plugging into the given equation, we get $36z^2 = 4x^2 + 9y^2$. Dividing by 36 we get $z^2 = \frac{x^2}{9} + \frac{y^2}{4}$ which is an elliptic cone with vertex at (0,0,0). In spherical co-ordinates the equation is obtained by replacing r/z with $tan\phi$ getting $36 = \frac{r^2}{z^2}(4cos^2\theta + 9sin^2\theta) = tan^2\phi(4cos^2\theta + 9sin^2\theta)$.

Soln for (b):

Plugging in $x = 3tcos\alpha, y = 2tsin\alpha, z = t$ into the equation $36z^2 = 4x^2 + 9y^2$ we get $36t^2 = 4(3tcos\alpha)^2 + 9(2tsin\alpha)^2 = 36t^2(cos^2\alpha + sin^2\alpha) = 36t^2$. This shows that every point on this curve also lies on the surface.

3. (20 points) Find the parametric equation of the line tangent to the graph of $\mathbf{r}(t) = e^t\mathbf{i} + cost\mathbf{j} + t^2\mathbf{k}$ at the point where $t = 0$.

Soln:

We have $\mathbf{r}'(t) == e^t\mathbf{i} - sint\mathbf{j} + 2t\mathbf{k}$. Then a vector along the tangent line is given by $\mathbf{r}'(0) = e^0\mathbf{i} - sin0\mathbf{j} + 2(0)\mathbf{k} = \mathbf{i}$.

The point given by $\mathbf{r}(0) = e^0\mathbf{i} + cos0\mathbf{j} + 0^2\mathbf{k} = \mathbf{i} + \mathbf{j}$ is a point on the tangent line.

So the equation of the tangent line can be written as $\mathbf{r}(t) = \mathbf{r}(0) + t\mathbf{r}'(0) = \mathbf{i} + \mathbf{j} + t\mathbf{i}$.

This can also be written as $x = 1 + t, y = 1, z = 0$.

4(a). (16 points) Find the arc length parametrization of the curve $\mathbf{r}(t) =< e^t, -e^t >$.

This parametrization must have $\mathbf{r}(0)$ as the reference point.

4(b). (9 points) For the curve in part (a) find the unit tangent at $s = 0$ and the curvature at any point. Why can't you find the unit normal at $s = 0$? What is special about this curve that makes it so?

Soln for 4(a):

The arc length parametrization is found by computing s in terms of t and then solving for t.

$$s(t) = \int_0^t ||\frac{d\mathbf{r}}{du}||\ du = \int_0^t \sqrt{x'(u)^2 + y'(u)^2}\ du$$
$$= \int_0^t \sqrt{(e^u)^2 + (-e^u)^2}\ du$$

Upon simplification you get $s(t) = \int_0^t e^u \sqrt{2}\ du = \sqrt{2}(e^t - 1)$.

Solving for t we get $e^t = \frac{s}{\sqrt{2}} + 1$ and $t = ln(\frac{s}{\sqrt{2}} + 1)$.

[We don't really need to solve for t in this particular case because the equation is given in terms of e^t].

So the arc-length parametrization obtained by plugging in s for t in $\mathbf{r}(t)$ (or in this case, s for e^t) is

$\mathbf{r}(s) =< \frac{s}{\sqrt{2}} + 1, -(\frac{s}{\sqrt{2}} + 1) >$

Solution for (4b):

The unit tangent at $s = 0$ is $\mathbf{T}(0) = \mathbf{r}'(0) =< 1/\sqrt{2}, -1/\sqrt{2} >$.

$\mathbf{T}'(s) = \mathbf{r}''(s) =< 0, 0 >$. So the curvature at any point given by $\kappa = ||\mathbf{r}''(s)|| = 0$.

This is not surprising because under the arc length parametrization, this curve is a line!!

In fact we have $\mathbf{r}(s) =< 1, -1 > + s < \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} >$.

The unit normal at $s = 0$ is undefined because $\mathbf{r}''(0) = 0$. If you plug it into the formula $\mathbf{N}(0) = \mathbf{r}''(0)/||\mathbf{r}''(0)||$ you get $\frac{0}{0}$. The reason is that there is no unique unit normal for the line. The unit normal $\mathbf{N}$ points inward towards the concave side of the curve, but for a line there is no such thing as a concave or convex side.

5. (Challenge, 20 points) Show that in cylindrical coordinates a curve given by the parametric equations $r = r(t), \theta = \theta(t), z = z(t)$ for $a \leq t \leq b$ has arc length

$$L = \int_a^b \sqrt{\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\ dt$$

Soln:

We have the arc length formula in rectangular coordinates given by:

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\ dt$$

Now plug in the cylindrical coordinates into this.
Keep in mind that r, θ, z are ALL $functions\ of\ t$!!
Using $x(t) = r(t)cos\theta(t)$ and $y(t) = r(t)sin\theta(t)$ in the formula for arc length we get

$$L = \int_a^b \sqrt{\left(\frac{dr}{dt}cos\theta - rsin\theta\frac{d\theta}{dt}\right)^2 + \left(\frac{dr}{dt}sin\theta + rcos\theta\frac{d\theta}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\ dt$$

Upon simplification this gives the desired formula.