2/4/2011 Spring 2011, Calculus III Test I Sitaraman

Instructions:
PLEASE PROVIDE STEP BY STEP EXPLANATIONS
ANSWERS WITHOUT EXPLANATION WILL ONLY GET 40 percent
Time Limit 50 minutes ; Total 100 points
Please read the questions carefully before answering
It is recommended that you try those problems you are most comfortable with, first.
Attempt as many as you can; Anything over 100 is extra credit.

1. (30 points) Say whether each statement is true or false. In each case, explain why it is true/false or give counterexample. If true it is NOT enough to give just one example. You may quote theorems from the book in support of your argument.

(a) Each line in 3-space is perpendicular to only one line.

(b) The intersection of a plane and a surface is always a curve.

(c) The sum of two vectors $\mathbf{u}$ and $\mathbf{v}$ will always be in the same plane as $\mathbf{u}, \mathbf{v}$.

Soln:

1a) FALSE. Actually every line will be perpendicular to a whole plane, containing infinitely many lines.

1b) False. The sphere with center at (0,0,1) and radius 1 will only touch the xy plane at exactly one point, namely (0,0,0).

1c) True.

$\mathbf{u} \times \mathbf{v}$ is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$.

Now $(\mathbf{u} \times \mathbf{v}).(\mathbf{u} + \mathbf{v}) = (\mathbf{u} \times \mathbf{v}).\mathbf{u} + (\mathbf{u} \times \mathbf{v}).\mathbf{v} = 0 + 0 = 0$ as well. So $\mathbf{u} \times \mathbf{v}$ is also perpendicular to $\mathbf{u} + \mathbf{v}$. This means $\mathbf{u} + \mathbf{v}$ has to be on same plane as $\mathbf{u}, \mathbf{v}$ because they are all perpendicular to the same vector $\mathbf{u} \times \mathbf{v}$.

2(a). (15 points) Find the volume of the parallelopiped that is formed by the vectors $< 1, 1, 1 >, < 1, 2, 0 >$ and $< -1, 0, 2 >$.

2(b). (10 points) Would it be possible to write $< 1, 1, 1 >$ as a linear combination of $< 1, 2, 0 >$ and $< -1, 0, 2 >$?

Soln:

2a) The volume of the parallelopiped is given by the magnitude of the scalar triple product of the three vectors.

The cross product of the vectors $< 1, 2, 0 >$ and $< -1, 0, 2 >$ is given by the determinant

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 0 \\ -1 & 0 & 2 \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 0 \\ -1 & 2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 2 \\ -1 & 0 \end{vmatrix} \mathbf{k} = 4\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}.$$

Taking its dot product with $< 1, 1, 1 >$ we get 4-2+2=4.

The magnitude of the scalar triple product is 4. So the volume of the parallelopiped formed by them is also 4.

2b) No, because they are not in the same plane (Scalar triple product is NOT zero!). You can show, using the same kind of argument as in 1c, that any linear combination $x\mathbf{u} + y\mathbf{v}$ of two vectors $\mathbf{u}, \mathbf{v}$ will be in the same plane as the two vectors $\mathbf{u}$ and $\mathbf{v}$.

Indeed, let $< 1,1,1 >= \mathbf{u}$ and the other two be $\mathbf{v}$ and $\mathbf{w}$. If $\mathbf{u} = k\mathbf{v} + l\mathbf{w}$ for some real numbers k, l then take the dot product with $\mathbf{v} \times \mathbf{w}$ on both sides. We get $4 = \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (k\mathbf{v} + l\mathbf{w}) \cdot (\mathbf{v} \times \mathbf{w}) = k\mathbf{v} \cdot (\mathbf{v} \times \mathbf{w}) + l\mathbf{w} \cdot (\mathbf{v} \times \mathbf{w}) = k(0) + l(0) = 0$. So we get $4 = 0$!! [Here we used the fact that $\mathbf{v} \times \mathbf{v} = 0$ and $\mathbf{w} \times \mathbf{w} = 0$ to show that the two scalar triple products are zero. For example, $\mathbf{w} \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{v} \cdot (\mathbf{w} \times \mathbf{w})$ – both are in same order].

3. (20 points) Find the equation of the plane passing through (1,0,1), (1,-1,0) and (2,3,5).

Soln:

Let $P = (1, 0, 1), Q = (1, -1, 0)$ and $R = (2, 3, 5)$.

To get the equation of the plane containing P, Q, R we need one point on the plane $\mathbf{r}_0$ and a vector $\mathbf{n}$ perpendicular to the plane. We can choose any one of the three points as $\mathbf{r}_0$. Let $P = \mathbf{r}_0$.

The normal vector can be chosen to be $\vec{PQ} \times \vec{PR}$. First note that $\vec{PQ}$ and $\vec{PR}$ are both in the same plane as P, Q, R. Secondly, being the cross product, it is perpendicular to both $\vec{PQ}$ and $\vec{PR}$. So the cross product is perpendicular to the plane of the points P,Q,R.

$\vec{PQ} = \vec{OQ} - \vec{OP} =< 1, -1, 0 > - < 1, 0, 1 >=< 0, -1, -1 >$.

Similarly $\vec{PR} =< 1, 3, 4 >$.

So $\mathbf{n} =< 0, -1, -1 > \times < 1, 3, 4 >= -\mathbf{i} - \mathbf{j} + \mathbf{k}$.

So the equation of the plane is $(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{n} = 0$ which gives $\mathbf{r} \cdot \mathbf{n} = \mathbf{r}_0 \cdot \mathbf{n}$.

Plugging in $\mathbf{r} =< x, y, z >, \mathbf{r}_0 = \vec{OP} =< 1, 0, 1 >, \mathbf{n} =< -1, 1, 1 >$

we get $-x - y + z = -1 + 1 = 0$. So the equation is $x + y - z = 0$.

Check that all three points satisfy this equation.

4(a). (15 points) By completing the square, write down the equation for the following quadric surface in a suitable way and identify it: $x^2 + 4x + 2y^2 - 4y = z$.

(b). (10 points) Describe the trace of the above surface when $x = 0, y = 0$ and $z = 0$. [These are the intersections with the xy plane, xz plane and yz plane].

Soln: First gather x, y and z terms separately.

You get $(x^2 + 4x) + 2(y^2 - 2y) = z$.

Completing the square, get $(x + 2)^2 - 2^2 + 2[(y - 1)^2 - 1^2] = z$.

Simplifying, we get $(x+2)^2 - 2^2 + 2[(y-1)^2 - 1^2] = z$ or $(x+2)^2 + 2(y-1)^2 = z+6$.

This can be written as $X^2 + 2Y^2 = Z$ with $X = x + 2, Y = y - 1, Z = z + 6$. It is an elliptic paraboloid in new co-ordinates X, Y, Z with vertex translated to (-2,1,-6).

You can see that it is a paraboloid (obtained by "revolving" a parabola about the z axis) by looking at the $X = 0, Y = 0$ and $Z = 0$ planes. The first two result in parabolas but the last one gives a point. There is no trace if Z is negative because the left hand side being comprised of squares cannot be negative. But if Z is positive, say $Z = 2$ for example, then you get an ellipse $X^2 + 2Y^2 = 2$ which is same as $X^2/2 + Y^2 = 1$. This is an ellipse with semi-major axis along X of length $\sqrt{2}$, the semi-minor axis along Y of length 1, center at $X = 0, Y = 0, Z = 2$. So it is an elliptic paraboloid facing up.

(b) This gives the traces made with the original curve, instead of the translated curve.

When $z = 0$ we get the trace on the xy plane. It is $x^2 + 4x + 2y^2 - 4y = 0$ which can be simplified to $(x + 2)^2 + 2(y - 1)^2 = 6$ which can be written as $\frac{(x+2)^2}{6} + \frac{(y-1)^2}{3} = 1$. This is an ellipse with center at (-2,1,0) and semi-major axis of length $\sqrt{6}$ along x axis and semi-minor axis of length $\sqrt{3}$ along the y axis.

When $x = 0$ we get $z = 2y^2 - 4y = 2(y - 1)^2 - 2$ or $z + 2 = 2(y - 1)^2$. This is a parabola with vertex at (0,1,-2).

Similarly when $y = 0$ we get a parabola with vertex (-2,0,-4).

5. (Challenge, 20 points) Using the equation of a plane, show that the plane that is tangent to the sphere $x^2 + y^2 + z^2 = r^2$ at the point $(\frac{r}{\sqrt{2}}, \frac{r}{\sqrt{2}}, 0)$ is perpendicular to the radial vector $< \frac{r}{\sqrt{2}}, \frac{r}{\sqrt{2}}, 0 >$.

Soln:

There is only one plane that will pass through a point (a,b,c), perpendicular to the vector $< a, b, c >$. That plane is given by the equation $ax + by + cz = a^2 + b^2 + c^2$ because $\mathbf{n} =< a, b, c >$ and $\mathbf{r}_0 =< a, b, c >$ as well. [The right hand side is $\mathbf{r}_0 \cdot \mathbf{n}$]. But since (a,b,c) is on the sphere, we must have $a^2 + b^2 + c^2 = r^2$ and hence $ax + by + cz = r^2$. Plugging in $a = \frac{r}{\sqrt{2}}, b = \frac{r}{\sqrt{2}}, c = 0$ we get $\frac{r}{\sqrt{2}}x + \frac{r}{\sqrt{2}}y = r^2$ which simplifies to $x + y = \sqrt{2}r$. This is a plane parallel to the $z-$axis, intersecting $xy-$plane along $x + y = \sqrt{2}r$.

It is enough, therefore, to show that this plane is tangent to the sphere. To find the intersection of this plane with the sphere, we plug in $x = \sqrt{2}r - y$ into the equation of the sphere and get $(\sqrt{2}r - y)^2 + y^2 + z^2 = r^2$. This simplifies to $r^2 - 2\sqrt{2}ry + 2y^2 + z^2 = 0$.

Now this gives equation can be thought of as a quadratic equation in y. Then using quadratic formula for $Ay^2 + By + C = 0$ with $A = 2, B = -2\sqrt{2}r, C = r^2 + z^2$ we get

$$y = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} = \frac{-(-2\sqrt{2}r) \pm \sqrt{(-2\sqrt{2}r)^2 - 4(2)(r^2 + z^2)}}{2(2)}$$

$$= \frac{2\sqrt{2}r \pm \sqrt{8r^2 - 8r^2 - 8z^2}}{4} = \frac{2\sqrt{2}r \pm \sqrt{-8z^2}}{4}$$

This gives real solutions only if $z = 0$. In that case we get just one solution $y = r/\sqrt{2}$. Plugging in y in $x + y = \sqrt{2}r$ (the equation for the plane) we get $x = r/\sqrt{2}$. So there is a unique solution for the intersection of the given plane and the sphere, namely $(\frac{r}{\sqrt{2}}, \frac{r}{\sqrt{2}}, 0)$ This shows that the plane is tangent to the sphere at $(\frac{r}{\sqrt{2}}, \frac{r}{\sqrt{2}}, 0)$ as required.