4/22/2011 Spring 2011, Calculus III Quiz 9 Sitaraman

Instructions:
PLEASE PROVIDE STEP BY STEP EXPLANATIONS
ANSWERS WITHOUT EXPLANATION WILL ONLY GET 40 percent
Time Limit 30 minutes ; Total 50 points
Please read the questions carefully before answering
It is recommended that you try those problems you are most comfortable with, first.
Attempt as many as you can; Anything over 50 is extra credit.

1. (10 points) Say whether each statement is true or false. If true prove your statement. [It is NOT enough to give just one example]. Otherwise, prove that it is false or provide a counterexample.
If a vector field $\mathbf{F}$ is the gradient of a function $f(x, y, z)$ all of whose second partial derivatives are continuous at every point (x, y, z) in a region S then $curl\mathbf{F} = \mathbf{0}$ at all points of S.

Solution.
True. This is actually exercise 38 in 15.1. We have

$$curl\mathbf{F} = curl\ \nabla f = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_x & f_y & f_z \end{vmatrix}$$

$$= (f_{zy} - f_{yz})\mathbf{i} + (f_{xz} - f_{zx})\mathbf{j} + (f_{yx} - f_{xy})\mathbf{k}$$

All the components of this vector zero because the $f_{xy} - f_{yx} = 0$ and so on. The equality of the mixed second partial derivatives is true because the second partial derivatives are all given to be continuous.

2. (20 points) For $\mathbf{F} = e^{xy}\mathbf{i} - cosy\mathbf{j} + sin^2z\mathbf{k}$ find $div\mathbf{F}$, $curl\mathbf{F}$ and $div(curl\mathbf{F})$.

Soln: This is exercise 20 in 15.1. As discussed in class, $div(curl)$ is always zero.

$$div\mathbf{F} = (e^{xy})_x + (-cosy)_y + (sin^2z)_z = ye^{xy} + siny + 2sinzcosz.$$

$$curl\ \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \mathbf{e}^{xy} & -cosy & sin^2 z \end{vmatrix} = 0\mathbf{i} + 0\mathbf{j} + (0 - xe^{xy})\mathbf{k} = -xe^{xy}\mathbf{k}.$$

3. (20 points) Evaluate the integral $\oint_C xdy - ydx$ where C is the circle $x^2 + y^2 = 1$ first using a parametric equation for the circle and then using Green's theorem.

Soln:

The parametric equation is $x = cost, y = sint$ with $0 \leq t \leq 2\pi$.

Plugging these equations into the integral , we get

$$\oint_C xdy - ydx = \int_0^{2\pi} cost(costdt) - sint(-sintdt) = \int_0^{2\pi} dt = 2\pi.$$

Note: If you want $\mathbf{F} \cdot d\mathbf{r} = xdy - ydx$ then you need $\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$.

For using Green's theorem, we first write $xdy - ydx$ as $\mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} =< -y, x >$. The circle is simply connected with a simple closed boundary and the functions $-y$ and x satisfy the desired continuity and differentiability conditions.

According to green's theorem, we get

$$\oint_C xdy - ydx = \int\int \left(\frac{\partial x}{\partial x} - \frac{\partial(-y)}{\partial y} \right) dA = \int\int 2dA = 2\pi.$$

The last integral equals 2π because $\int\int dA$ over the circle is just the area of the circle of radius 1.

4. (Bonus, 15 points) Find the work done by the force of gravity (assume $\mathbf{F} = mg$ with $g = 32feet/sec^2$) while moving body of mass 10 pounds along an inclined plane of length 100 feet at 45 degrees to the horizontal by using a line integral. Then compute the same using the formula Work = Force times distance.

Soln:

The inclined plane has slope $tan45 = 1$. The height of the inclined plane is given by $sin(45) = h/100$ which means $h = 100(1/\sqrt{2}) = 70.71feet$. Since slope is 1, the rise equals the run and thus the top of the inclined plane will be at (70.71,70.71) if the bottom is at (0,0). This is a line with equation $y = x$. The equation is $\mathbf{r} =< x, y >$ .The force is given by $(0, -320)$ because gravity is along the negative $y-$direction. Note that the body is going from the top to the bottom. So the integral is

$$\int_{70.71}^{0} < 0, -320 >< dx, dy >= \int_{70.71}^{0} -320dy = 320(70.71) = 22627.2.$$

The force is measured in lb-feet/sec/sec.

Using the usual formula we first need to find the force **along** the inclined plane. This is given by $\mathbf{F}cos45 = 320(\frac{1}{\sqrt{2}})$. This times distance equals $320(100/\sqrt{2}) = 22627.2$ lb-feet/sec/sec.