4/15/2011 Spring 2011, Calculus III Quiz 8 Sitaraman

Instructions:
PLEASE PROVIDE STEP BY STEP EXPLANATIONS
ANSWERS WITHOUT EXPLANATION WILL ONLY GET 40 percent
Time Limit 30 minutes ; Total 50 points
Please read the questions carefully before answering
It is recommended that you try those problems you are most comfortable with, first.
Attempt as many as you can; Anything over 50 is extra credit.

1. (10 points) Say whether each statement is true or false. If true prove your statement. [It is NOT enough to give just one example]. Otherwise, prove that it is false or provide a counterexample.

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{0}^{\sqrt{4-x^2-y^2}} z^2\sqrt{x^2+y^2+z^2}\ dzdydx$$

$$=\int_{0}^{2\pi}\int_{0}^{\pi/2}\int_{0}^{2}\rho^5 cos^2\phi sin\phi d\rho d\phi d\theta$$

Solution.

True.This is worked out in example 4, section 14.6. The solid of integration is just the upper hemisphere of a sphere of radius 2.

2. (20 points) Find the volume of the solid bounded by the surface $z = x^2 + y^2$ and the planes $x = y, y = 0, z = 0$ and $x = 1$. Use triple integrals and rectangular (or cylindrical) co-ordinates.[Note: Cylindrical is harder!]

Soln: This is a solid in the first octant (x, y, z are all positive).

In the xy plane under the solid, the region of integration is bounded by the $x-$axis ($y = 0$), the line $x = y$ and the line $x = 1$.

As x goes from 0 to 1, y goes from 0 to x.

The top of the solid is the part of the paraboloid $z = x^2 + y^2$ in the first octant.

So z goes from 0 to $x^2 + y^2$.

So the integral is

$$\int_0^1 \int_0^x \int_0^{x^2+y^2} dzdydx = \int_0^1 dx \int_0^x (x^2+y^2)dy = \int_0^1 \left[x^2y + \frac{y^3}{3}\right]_0^x dx$$

$$= \int_0^1 \frac{4x^3}{3} dx = \frac{1}{3}.$$

In cylindrical coordinates:

The angle θ in the (r, θ) plane goes from 0 to $\pi/4$ as the polar ray sweeps from the x-axis to the line $x = y$ which makes an angle 45 degrees or $\pi/4$ radians with $x-$axis.

For each θ, r goes from 0 to $sec\theta$ because r goes from 0 to the line $x = 1$ where it is given by $rcos\theta = 1$.

For each (r, θ) the z value goes from 0 to $x^2 + y^2 = r^2$.

So the integral in cylindrical coordinates is

$$\int_0^{\pi/4} \int_0^{sec\theta} \int_0^{r^2} r\ dzdrd\theta = \int_0^{\pi/4} d\theta \int_0^{sec\theta} r[z]_0^{r^2}\ dr = \int_0^{\pi/4} \frac{sec^4\theta}{4} d\theta$$

$$= \frac{1}{4}\left[\frac{sec^2\theta tan\theta}{3} + \frac{2}{3}\int sec^2\theta d\theta\right]_0^{\pi/4} \quad [formula(20), 7.3, p.503]$$

$$= \frac{1}{4}[\frac{2}{3} + \frac{2}{3}] = \frac{1}{3}.$$

3. (20 points) Evaluate the following integral:

$$\int_0^{\pi/2} \int_0^{cos\theta} \int_0^{r^2} rsin\theta\ dzdrd\theta$$

(10 points bonus) Describe the solid over which this integral is evaluated.

Soln:

$$\int_0^{\pi/2} \int_0^{cos\theta} \int_0^{r^2} rsin\theta\ dzdrd\theta = \int_0^{\pi/2} sin\theta d\theta \int_0^{cos\theta} rdr \int_0^{r^2} dz$$

$$= \int_0^{\pi/2} sin\theta d\theta \int_0^{cos\theta} r(r^2)dr = \int_0^{\pi/2} sin\theta d\theta (\frac{cos^4\theta}{4})$$

Letting $u = cos\theta$ this integral becomes

$$\int_0^{\pi/2} sin\theta d\theta (\frac{cos^4\theta}{4}) = \int_1^0 \frac{u^4}{4}(-du) = \int_0^1 (u^4/4)du = \frac{1}{20}.$$

The solid is the part of the paraboloid $z = r^2$ in the first octant that lies over the circle $r = cos\theta$ which has center at (1/2,0) and radius 1/4.

THIS PROBLEM IS EASY!! 4. (Bonus, 15 points) Use parametric equations to show that the surface area of a sphere of radius R is $4\pi R^2$.

Soln:

Let $u = x, v = y$. Then $z = f(u,v) = f(x,y) = \sqrt{R^2 - x^2 - y^2}$ and

$$||\frac{\partial \mathbf{r}}{\partial x} \times \frac{\partial \mathbf{r}}{\partial y}|| = \sqrt{(\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2 + 1}$$

$$= \sqrt{\frac{x^2}{R^2 - x^2 - y^2} + \frac{y^2}{R^2 - x^2 - y^2} + 1} \quad = \frac{R}{\sqrt{R^2 - x^2 - y^2}}$$

[Note: if you want to convert $\sqrt{(\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2 + 1}$ to polar coordinates here then you must use

$$\left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2$$

Note that there is a $1/r^2$ term multiplied with $\left(\frac{\partial z}{\partial \theta}\right)^2$. This is worked out in exercise 55 of 13.5].

So the surface area is given by the integral

$$2\int_{-R}^{R} \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \frac{R}{\sqrt{R^2 - x^2 - y^2}} dydx$$

[Need to multiply by 2 to get both upper and lower hemispheres]

This is better evaluated using polar coordinates. θ will go from 0 to 2π and for each θ, r will go from 0 to R. We will get

$$2\int_0^{2\pi}\int_0^R \frac{R}{\sqrt{R^2-r^2}} r dr d\theta = R\int_0^{2\pi} d\theta \int_0^R \frac{2r dr}{\sqrt{R^2-r^2}}$$

$$= R\int_0^{2\pi} d\theta \int_0^{R^2} \frac{du}{\sqrt{u}} = R\int_0^{2\pi} d\theta \left[2\sqrt{u}\right]_0^{R^2} = 4\pi R^2.$$