4/8/2011 Spring 2011, Calculus III Quiz 7 Sitaraman

Instructions:
PLEASE PROVIDE STEP BY STEP EXPLANATIONS
ANSWERS WITHOUT EXPLANATION WILL ONLY GET 40 percent
Time Limit 30 minutes ; Total 50 points
Please read the questions carefully before answering
It is recommended that you try those problems you are most comfortable with, first.
Attempt as many as you can; Anything over 50 is extra credit.

1. (10 points) Say whether each statement is true or false. If true prove your statement. [It is NOT enough to give just one example]. Otherwise, prove that it is false or provide a counterexample.
a. $\int\int f(x,y)\ dxdy = \int\int f(x,y) dydx$ over a region R if $f(x,y)$ is differentiable over all of R.
b. The volume of the sphere of radius R whose equation is
$x^2 + y^2 + z^2 = R^2$ is given by

$$4\int_0^R \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \sqrt{R^2 - x^2 - y^2}\ dydx.$$

Solution for 1a.

True. If $f(x,y)$ is differentiable then it is continuous. If it is continuous then order of integration doesn't matter, according to Fubini's theorem (Theorem 14.1.3).

Solution for 1b.

True. This is the correct formula for finding the volume of sphere. The volume is calculated by multiplying the volume of one half of the upper hemisphere (the half where x is positive, z is positive and y can be positive and negative) by 4. The region of integration on the $xy-$plane is the half of the circle $x^2 + y^2 = R^2$ where x is positive. We integrate over y first, and for each x the value of y ranges from $-\sqrt{R^2 - x^2}$ to $\sqrt{R^2 - x^2}$ because the boundary of the region is given by $x^2 + y^2 = R^2$. then x ranges from 0 to R. The value of z for each (x, y) is given by $\sqrt{R^2 - x^2 - y^2}$ so this is the function to be integrated, to get the volume of the sphere.

2. (20 points) After correcting the formula in 1b if necessary, use it to show that the volume of the sphere is $\frac{4\pi R^3}{3}$.

Soln: Evaluating the inside integral (the one that is with respect to y) using the substitution $\sqrt{R^2 - x^2} sin\theta = y$ we get
$\int \sqrt{R^2 - x^2} cos\theta (\sqrt{R^2 - x^2} cos\theta) d\theta = (R^2 - x^2) \int cos^2\theta \; d\theta$.
[We can take out the $R^2 - x^2$ because it is independent of y].

Using the formula $cos^2\theta = (1 + cos2\theta)/2$ or otherwise, we evaluate this integral to be $(\theta + (sin2\theta/2))/2$. Now if we put $\sqrt{R^2 - x^2} sin\theta = y$ then the limits of the inside integral change to $-\pi/2$ and $\pi/2$ respectively. Plugging in the limit values for θ we get the inside integral to be $(R^2 - x^2)\pi/2$. Integrating this with respect to x and plugging in the limits 0 and R you get that the volume is $\frac{4}{3}\pi R^3$.

You are really supposed to do this in rectangular coordinates but since the problem didn't expressly say so, I gave full credit if you did it using polar co-ordinates

Soln in polar: The integral becomes

$$4 \int_0^{\pi} \int_0^{R} \sqrt{R^2 - r^2} \; r dr d\theta$$

Evaluating using the substitution $u = R^2 - r^2$ we will get $\frac{4}{3}\pi R^3$.

3. (20 points) Show that the volume under the cone $z^2 = x^2 + y^2$, inside the cylinder $x^2 + y^2 = 1$ and above $z = 0$ is

$$4 \int_0^1 \int_0^{\sqrt{1-x^2}} (\sqrt{x^2 + y^2}) \; dy dx$$

Calculate this **after converting to polar co-ordinates** and show that this equals $\frac{2}{3}\pi$. Prove this geometrically by subtracting the volume of the cone of radius 1 and height 1 from the volume of the cylinder of same dimensions [The general formula for volume of cone is $\frac{1}{3}\pi R^2 H$ where R is the radius of the base and H is the height].

Soln:

The base is given by $z = 0 \Rightarrow x^2 + y^2 = 1$ and the value of z on the surface is given by $z = \sqrt{x^2 + y^2}$. Integrating over the first quadrant we have x going from 0 to 1 and y going from 0 to $\sqrt{1-x^2}$ for a given x. Since function is symmetrical we can get entire volume by multiplying volume over first quadrant by 4.

In polar coordinates their equations are $r = 1$ and $z = r$.

The integral is

$$4\int_0^{\pi/2}\int_0^1 (r)\ rdrd\theta = 4\int_0^{\pi/2}\int_0^1 r^2 dr\ d\theta == 4\int_0^{\pi/2}\left[\frac{r^3}{3}\right]_0^1\ d\theta = \frac{2\pi}{3}$$

Geometrically, volume of cylinder is $\pi R^2 H = \pi$. Volume of cone is $\frac{1}{3}\pi R^2 H = \frac{1}{3}\pi$. So the volume of the solid inside the cylinder, outside the cone is $\frac{2\pi}{3}$.

THIS PROBLEM IS EASY!! 4. (Bonus, 15 points) Suppose that the temperature in degrees celsius at a point (x, y) on a flat metal plate is $T(x, y) = 10 - 8x^2 - 2y^2$ where x and y are in meters. Find the average temperature and the maximum and minimium temperatures on the plate for which $0 \leq x \leq 1$ and $0 \leq y \leq 2$.

Soln:

This is problem 37 of 14.1. The average temperature is obtained by integrating the temperature over the whole rectangle and dividing by the area which is 2 (length is 1 and width is 2).

So we get

$$T_{average} = \frac{1}{2}\int_0^1\int_0^2 10 - 8x^2 - 2y^2\ dydx$$

$$= \frac{1}{2}\int_0^1 \left[10y - 8x^2y - 2(y^3/3)\right]_0^2\ dx = \frac{1}{2}\int_0^1 20 - 16x^2 - 2(8/3)\ dx$$

$$= \frac{1}{2}\int_0^1 \frac{44}{3} - 16x^2\ dx = \frac{1}{2}[(44/3) - (16/3)] = \frac{14}{3}^\circ C$$

The rectangle is a closed and bounded set, so T should attain max and min on it. $T_x = -16x$ and $T_y = -4y$. Setting them equal to 0 we get that the only critical point is (0,0). Note that (0,0) is not in the interior, it is on the boundary.

On the boundary $x = 0$, we have $T = 10 - 2y^2$ and $T' = -4y$ and its critical point is also $(0, 0)$. On $x = 1$, we have $T = 2 - 2y^2$ and again the critical point is at $y = 0$. This gives (1,0) as another critical point (albeit on the boundary). On $y = 0$ again we end up with $(0, 0)$ and on $y = 1$ also we end up with (0,1).

Now we compare the values at the three critical points:

At (0,0) we get $T = 10$.

At (0,1) we get $T = 8$.

At (1,0) we get $T = 2$.

So the temperature is absolute minimum at (1,0) at $2°C$ and absolute maximum on (0,0) at $10°C$.