1. (10 points) Say whether each statement is true or false. In each case, explain why it is true/false or give counter-example. If true it is NOT enough to give just one example.
(a) If $f(x,y,z) = f(x(t),y(t),z(t))$ then $\frac{df}{dt} = \frac{df}{dx}\frac{dx}{dt} + \frac{df}{dy}\frac{dy}{dt} + \frac{df}{dz}\frac{dz}{dt}$.
[Note: $\frac{d}{dx}$ is not same as $\frac{\partial}{\partial x}$].
(b)If $\nabla F = 0$ for $F(x,y,z)$ at a point P, then at P the derivative of F along any vector $\mathbf{u}$ is zero.

Soln:
1a) FALSE. One should have $\frac{\partial f}{\partial x}$ etc instead of $\frac{df}{dx}$ etc.,
1b) True, because the directional derivative along any vector is a multiple of the scalar product of the gradient with that vector. Since gradient is given to be 0 and scalar product of any vector with the zero vector is zero, the directional derivative also is zero. [In fact it is equal to $\nabla F \cdot \mathbf{u}_1$ where $\mathbf{u}_1$ is the unit vector along $\mathbf{u}$].

2(a). (10 points) Given $f(x,y) = x^2 + y^2$ and $x = u + v, y = u - v$ find f_u, f_v in terms of u, v.
2(b). (10 points) Using the f in (a), Write $f(x,y) = 1$ as $g(u,v) = 1$ by substituting for x, y with u, v as given. From this equation find $\frac{du}{dv}$ using implicit differentiation.

Soln:
2a) $f_u = f_x x_u + f_y y_u = (2x)(1) + (2y)(1) = 2x + 2y = 4u$ and
$f_v = f_x x_v + f_y y_v = (2x)(1) + (2y)(-1) = 2x - 2y = 4v$.
2b) Plugging in $x = u+v, y = u-v$ we get $(u-v)^2+(u+v)^2 = 2u^2+2v^2 = 1$. So $g(u,v) = 2u^2 + 2v^2$. Then $\frac{du}{dv} = -g_v/g_u = -4v/(4u) = -v/u$.
Notice that this is same as $-f_v/f_u$. That is because f and g are really the same when considered as functions of u, v.

3. (20 points) Describe the intersection of the sphere $x^2 + y^2 + z^2 = 2$ with the cone $z^2 = x^2 + y^2$. Find the equation of the tangent planes of the two surfaces at $(0,1,1)$. Show that, in fact, the tangent planes of the two surfaces are perpendicular to each other at all the points of intersection.

Soln:

The intersection is given by combining the two equations and getting the common solutions. Plugging in $z^2 = x^2 + y^2$ into the first equation we get $z^2 + z^2 = 2$ which gives $z^2 = 1$ or $z = \pm 1$. Thus the two equations intersect on the planes $z = 1$ and $z = -1$. But $z = \pm 1$ gives $x^2 + y^2 = 1$. Thus the points of intersection are on the circles of radius 1 centered on the $z-$axis and sitting in the $z = 1$ and $z = -1$ planes. [So the circles have centers at (0,0,1) and (0,0,-1)].

At (0,1,1) the normals to the planes are given by $\mathbf{n}_1 = \nabla F$ for the sphere and $\mathbf{n}_2 = \nabla G$ for the cone where $F = x^2 + y^2 + z^2$ and $G = z^2 - x^2 - y^2$.

So we get $\mathbf{n}_1 = \nabla F =< \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} >=< 2x, 2y, 2z >=< 0, 2, 2 >$ at (0,1,1).

And we have $\mathbf{n}_2 = \nabla G =< \frac{\partial G}{\partial x}, \frac{\partial G}{\partial y}, \frac{\partial G}{\partial z} >$

$=< -2x, -2y, 2z >=< 0, -2, 2 >$ at (0,1,1).

[Note that these two vectors are perpendicular –their dot product is zero].

The tangent plane to the sphere at (0,1,1) is given by $0(x-0) + 2(y-1) + 2(z-1) = 2y + 2z - 4 = 0$ which is same as $y + z = 2$.

The tangent plane to the cone at (0,1,1) is given by $0(x-0) - 2(y-1) + 2(z-1) = -2y + 2z = 0$ which is same as $y - z = 0$.

From these calculations we can see that at a general point of intersection $(x, y, \pm 2)$ we have $\mathbf{n}_1 =< 2x, 2y, \pm 2 >$ and $\mathbf{n}_2 =< -2x, -2y, \pm 2 >$.

But then $\mathbf{n}_1 \cdot \mathbf{n}_2 =< 2x, 2y, \pm 2 > \cdot < -2x, -2y, \pm 2 >= -4x^2 - 4y^2 + 4$. We also know the points of intersection satisfy $x^2 + y^2 = 1$. So upon multiplying by 4 we get $-4x^2 - 4y^2 + 4 = \mathbf{n}_1 \cdot \mathbf{n}_2 = 0$ for all these points! Thus the the tangent planes will be perpendicular at every point where the given sphere and the cone intersect.

4. (Challenge, 15 points) If $f(x,t) = \int_0^t e^{u^2 x} du$ show that f satisfies the equation $f(x,t) + 2x f_x = t e^{xt^2}$.

Soln:

Since the integral limits are independent of x and the integration itself is not over x we can differentiate under the integral sign.

[Use basic definition of partial derivative to see that f_x is the limit of $(1/\Delta x)[\int_0^t e^{u^2(x+\Delta x)} du - \int_0^t e^{u^2 x} du] = (1/\Delta x)[\int_0^t (e^{u^2(x+\Delta x)} - e^{u^2 x}) du]$.

Differentiating under the integral we get $f_x = \int_0^t u^2 e^{u^2 x} du$. Using integration by parts (with $dU = 2uxe^{u^2 x}$ and $V = u/2x$) we have $\int u^2 e^{u^2 x} du = (u/2x)e^{u^2 x} - \int \frac{e^{u^2 x}}{2x}\ du$. Plug everything back into the integral for f_x and evaluate at 0 and t to get $f_x = \frac{te^{xt^2}}{2x} - (1/2x)f(x,t)$. [$(1/2x)$ can be taken out of the second integral because the integration is over u and not x]. Simplify to get the desired equation.