2/18/2011 Spring 2011, Calculus III Quiz 4 Sitaraman

Instructions:
PLEASE PROVIDE STEP BY STEP EXPLANATIONS
ANSWERS WITHOUT EXPLANATION WILL ONLY GET 40 percent
Time Limit 30 minutes ; Total 50 points
Please read the questions carefully before answering
It is recommended that you try those problems you are most comfortable with, first.
Attempt as many as you can; Anything over 50 is extra credit.

1. (10 points) Say whether each statement is true or false. In each case, explain why it is true/false or give counter-example. If true it is NOT enough to give just one example.
(a) The integral $\int_a^b ||\mathbf{r}'(t)||dt$ gives the change in position $\mathbf{r}(b) - \mathbf{r}(a)$ according to the fundamental theorem of calculus.
(b) The curvature κ of a line is m where $\mathbf{r}(s) = \mathbf{r}_0 + s\mathbf{v}$ is the equation of the line with respect to the arc-length parameter s, and $\mathbf{v} =< m, m, m >$

Soln:
1a) FALSE. The integral $\int_a^b ||\mathbf{r}'(t)||dt$ actually gives $s(b) - s(a)$ according to the fundamental theorem, where $s(b) - s(a)$ is the arc length from a to b. To get the change in position vector you need to integrate $\mathbf{r}'(t)$.
1b) FALSE. The curvature of any line is 0 at all points.

2(a). (10 points) Find the arc length of the part of a curve given by $\mathbf{r}(t) =< e^t, e^{-t}, \sqrt{2}t >$ when $0 \le t \le 1$.
2(b). (10 points) Find the unit normal and tangent vectors $\mathbf{N}(0)$ and $\mathbf{T}(0)$ to the curve of part (a).
[Hint: May help to simplify $(e^t)^2 + (e^{-t})^2 + 2$ by writing it as a square].

Soln:
2a) The arc length is given by

$$\int_0^1 \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\ dt = \int_0^1 \sqrt{(e^t)^2 + (-e^{-t})^2 + (\sqrt{2})^2}\ dt$$

$$= \int_0^1 \sqrt{(e^t)^2 + (e^{-t})^2 + 2}\ dt = \int_0^1 \sqrt{(e^t + e^{-t})^2}\ dt = \int_0^1 (e^t + e^{-t})dt$$

$$= [e^t - e^{-t}]_0^1 = [e-1] - [\frac{1}{e} - 1] = e - (1/e) = 2.35.$$

2b)We have $\mathbf{r}'(t) =< e^t, -e^{-t}, \sqrt{2} >$. $||\mathbf{r}'(t)||$ was found in part (a) to equal $e^t + e^{-t}$. Now we know that $T(t) = \frac{\mathbf{r}'(t)}{||\mathbf{r}'(t)||}$. So all we need to do is to plug in $t = 0$ in this formula, using $\mathbf{r}'(t)$ and $||\mathbf{r}'(t)||$ as found above.

$$T(0) = \frac{\mathbf{r}'(0)}{||\mathbf{r}'(0)||} = \frac{< e^0, -e^0, \sqrt{2} >}{e^0 + e^0} =< \frac{1}{2}, -\frac{1}{2}, \frac{\sqrt{2}}{2} > .$$

Similarly, to find $\mathbf{N}(0)$ we use the formula $\frac{\mathbf{T}'(0)}{||\mathbf{T}'(0)||}$.
We have, using $\mathbf{r}'(t)$ and $||\mathbf{r}'(t)||$ as found above,

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{||\mathbf{r}'(t)||} =< \frac{e^t}{e^t + e^{-t}}, -\frac{e^{-t}}{e^t + e^{-t}}, \frac{\sqrt{2}}{e^t + e^{-t}} > .$$

Now we have to find $\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{||\mathbf{T}'(t)||}$ and then plug in 0.
First we find (using quotient rule frequently)

$$\mathbf{T}'(t) =< \frac{2}{(e^t + e^{-t})^2}, -\frac{e^{-2t}}{(e^t + e^{-t})^2}, \frac{-\sqrt{2}(e^t - e^{-t})}{(e^t + e^{-t})^2} >$$

Now $\mathbf{N}(0) = \frac{\mathbf{T}'(0)}{||\mathbf{T}'(0)||}$.
Plugging in 0 for t in the $\mathbf{T}'(t)$ we get
$\mathbf{T}'(0) =< \frac{1}{2}, -\frac{1}{4}, 0 >$ and its magnitude is $\sqrt{\frac{1}{4} + \frac{1}{16}} = \sqrt{5}/4$.
So $\mathbf{N}(0) =< \frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}}, 0 >$
[Remember to first differentiate and *then* plug in $t = 0$. Also, note how $\mathbf{T}(t)$ is a unit vector but $\mathbf{T}'(t)$ isn't].

3a. (6 points) In two dimensions, write the equation of the circle of radius 2 with center at (0,0) in parametric vector form. i.e, give an equation for $\mathbf{r}(t)$, the position vector of a point on the circle.
3b.(8 points) Find the arc length parametrization of the circle in 3a.
3c.(8 points) Using 3b, find the curvature for any value of s.

Soln:

3(a): The parametric vector equation is $\mathbf{r}(t) = 2cost\mathbf{i} + 2sint\mathbf{j}$ where t is measured in radians and t is usually chosen to be in $[0, 2\pi]$.

3(b): The arc length from 0 to t is $s(t) = \int_0^t \sqrt{x'(u)^2 + y'(u)^2}\ du = 2t$. So $t = s/2$ and the arc length parametrization is :

$\mathbf{r}(s) = 2cos(s/2)\mathbf{i} + 2sin(s/2)\mathbf{j}$.

3(c): The curvature

$$\kappa = ||\mathbf{r}''(s)|| = || - 2(1/4)cos(s/2)\mathbf{i} - 2(1/4)sin(s/2)\mathbf{j}|| = 1/2.$$

Notice that curvature is reciprocal of the radius.

4. (Challenge, 15 points) Find the parametric vector equation (i.e, equation for $\mathbf{r}(t)$) of the curve whose tangent vector satisfies

$\mathbf{r}'(t) = tsint\mathbf{i} + te^t\mathbf{j} + \mathbf{k}$ and which curve also satisfies $\mathbf{r}(0) = \mathbf{i}$.

Soln:

Let $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$.

$\mathbf{r}'(t) = tsint\mathbf{i} + te^t\mathbf{j} + \mathbf{k}$ means

$$x'(t) = tsint,\ y'(t) = te^t,\ z'(t) = 1.$$

Integrating, we get [Integrating by parts for first two equations]

$$x(t) = \int tsintdt = -tcost + sint + C_1, y(t) = te^t - e^t + C_2, z(t) = t + C_3.$$

Letting $t = 0$ we get

$$x(0) = C_1, y(0) = -1 + C_2, z(0) = C_3.$$

Writing this in vector form we get $\mathbf{r}(0) = C_1\mathbf{i} + (C_2 - 1)\mathbf{j} + C_3\mathbf{k}$.

It is given, though, that $\mathbf{r}(0) = \mathbf{i}$.

So we have $\mathbf{r}(0) = \mathbf{i} = C_1\mathbf{i} + (C_2 - 1)\mathbf{j} + C_3\mathbf{k}$.

Comparing coefficients and solving for the C's, we get $C_1 = 1, C_2 = 1, C_3 = 0$.

So after plugging in for the C's in the equation for $\mathbf{r}(t)$ we get $\mathbf{r}(t) = (-tcost + sint + 1)\mathbf{i} + (te^t - e^t + 1)\mathbf{j} + t\mathbf{k}$ as the required equation.