2/11/2011 Spring 2011, Calculus III Quiz 3 Sitaraman

Instructions:
PLEASE PROVIDE STEP BY STEP EXPLANATIONS
ANSWERS WITHOUT EXPLANATION WILL ONLY GET 40 percent
Time Limit 30 minutes ; Total 50 points
Please read the questions carefully before answering
It is recommended that you try those problems you are most comfortable with, first.
Attempt as many as you can; Anything over 50 is extra credit.

1. (10 points) Say whether each statement is true or false. In each case, explain why it is true/false or give counter-example. If true it is NOT enough to give just one example.
(a) In cylindrical co-ordinates, the equation of the sphere of radius 1, center at (0,0,0) is $r = 1$.
(b)The parametric equation $x = t, y = 0, z = 0$ represents the vector valued function $\mathbf{r}(t) = t\mathbf{i}$ and its graph is a plane.

Soln:

1a) FALSE. The equation $r = 1$ in cylindrical co-ordinates actually represents a cylinder of radius 1 and extending infinitely in both directions parallel to the $z-$axis, because there is no restriction on θ and ϕ. The equation for a sphere in cylindrical co-ordinates is $r^2 + z^2 = 1$. You get this by putting $r^2 = x^2 + y^2$ in $x^2 + y^2 + z^2 = 1$.

1b) FALSE. A vector valued function of one variable t will always have a curve for its graph. The graph of this equation in particular is simply the $x-$axis, since it passes through (0,0,0) and is parallel to the $\mathbf{i}$ vector.

2(a). (10 points) Write the equation $z = x^2 + y^2$ in cylindrical co-ordinates.
2(b). (10 points) Identify and graph this surface.

Soln:

2a) This is the equation of a (circular) paraboloid. Putting $x = rcos\theta, y = rsin\theta$ we get $z = r^2cos^2\theta + r^2sin^2\theta = r^2(cos^2\theta + sin^2\theta) = r^2$. So the equation in cylindrical co-ordinates is $z = r^2$.

2b) First note that the graph of this is only in the upper half of 3-space, above the $xy-$plane, because r^2 is always positive and thus z co-ordinate has to be positive for all points on the surface. Secondly, given any point (x, y, z) we get that all the points with the same z co-ordinate lie on the circle with radius r given by $r^2 = z$, or $r = \sqrt{z}$. So if you take cross-sections of this surface parallel to $xy-$plane at various values of z you get circles of radius $\sqrt{z}$. That is why it is a circular paraboloid. Finally, if you take cross sections vertically, say along $xz-$plane or $yz-$plane, we get parabolas: if $x = 0$ you get $z = y^2$ and if $y = 0$ you get $z = x^2$. So the vertical cross-sections are parabolas. The surface is obtained by rotating a parabola $z = x^2$ or $z = y^2$ about the $z-$axis.

3a. (10 points) Write the vector equation of the curve of intersection of the surfaces $z = x^2 + y^2$ and $x - y = 0$ in terms of a parameter t.
3b. (10 points) Find the equation for the tangent line to the curve of intersection in part (a) at (1,1,2).

Soln: (a):

Plugging in $x = y$ in the equation of the first surface, we get $z = 2x^2$ which is a parabola.

Letting $x = t$ we get $x = t, y = t, z = 2t^2$ as the parametric equation of the curve.

The vector equation is $\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + t^2\mathbf{k}$.

The tangent vector at (1,1,2) is $\mathbf{r}'(1)$ because $t = 1$ at (1,1,2).

Now $\mathbf{r}'(t) = \mathbf{i} + \mathbf{j} + 4t\mathbf{k}$.

So $\mathbf{r}'(1) = \mathbf{i} + \mathbf{j} + 4\mathbf{k}$.

The tangent line at (1,1,2) is therefore given by

$\mathbf{r}(t) =< 1, 1, 2 > +t < 1, 1, 4 >$.

4. (Challenge, 15 points) A bug is traveling along a circular helix

$\mathbf{r}(t) = cost\mathbf{i} + sint\mathbf{j} + t\mathbf{k}$ with t represnting time. At what points is it going straight up? If the velocity vector at time t is given by $\mathbf{r}'(t)$ find the point at which its speed is maximum.

Soln:

Velocity is given by $-sint\mathbf{i} + cost\mathbf{j} + \mathbf{k}$.

Its direction at a given time is given by its tangent vector which is also same as the velocity vector $-sint\mathbf{i} + cost\mathbf{j} + \mathbf{k}$..

If it is going straight up then tangent vector must be parallel to **k**.

This will happen only if $x = 0$ and $y = 0$ in the tangent vector.

That means $-sint = 0$ and $cost = 0$. But sine and cosine are never simultaneously zero. So this would never happen.

Speed is given by the magnitude of velocity which is $\sqrt{cos^2t + sin^2t + 1^2} = \sqrt{2}$. So its speed is always constant at $\sqrt{2}$ units.

NOTE: We haven't yet discussed a way to find the maximum of a *vector*. It makes sense only to look at the maximum of the *magnitude* of the vector.