1/28/2011 Springl 2011, Calculus III Quiz 2 Sitaraman

Instructions:
PLEASE PROVIDE STEP BY STEP EXPLANATIONS
ANSWERS WITHOUT EXPLANATION WILL ONLY GET 40 percent
Time Limit 30 minutes ; Total 50 points
Please read the questions carefully before answering
It is recommended that you try those problems you are most comfortable with, first.
Attempt as many as you can; Anything over 50 is extra credit.

1. (10 points) Say whether each statement is true or false. In each case, explain why it is true/false or give counterexample. If true it is NOT enough to give just one example.

(a) $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{c}) \cdot \mathbf{b}$.

(b) The points $P = (1, 1, 4), Q = (2, 2, -8)$ and $R = (3, 3, 12)$ lie on the same line.

Soln:

1a) FALSE. Actually $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = -(\mathbf{a} \times \mathbf{c}) \cdot \mathbf{b}$

First note that $\mathbf{b} \times \mathbf{c} = -\mathbf{c} \times \mathbf{b}$.

So $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = -\mathbf{a} \cdot (\mathbf{c} \times \mathbf{b})$.

But it is easy to see that $\mathbf{a} \cdot (\mathbf{c} \times \mathbf{b}) = (\mathbf{a} \times \mathbf{c}) \cdot \mathbf{b}$ by switching rows twice in the determinant for $\mathbf{a} \cdot (\mathbf{c} \times \mathbf{b})$.

[Or you can see that $\mathbf{a} \cdot (\mathbf{c} \times \mathbf{b}) = \mathbf{b} \cdot (\mathbf{a} \times \mathbf{c})$ because the order of the vectors is the same in both; But because dot product is commutative, $\mathbf{b} \cdot (\mathbf{a} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{c}) \cdot \mathbf{b}$].

Putting everything together we get that $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = -(\mathbf{a} \times \mathbf{c}) \cdot \mathbf{b}$.

1b)FALSE. The vector $\overrightarrow{PQ} = (2, 2, -8) - (1, 1, 4) = (1, 1, -12)$ can be easily to be not along the same line as $\overrightarrow{QR} = (1, 1, 20)$ because one is not a scalar multiple of another.

Another way to do this is to show that $\overrightarrow{PQ} \times \overrightarrow{QR}$ is not zero.

2(a). (12 points) Find the area of the parallelogram that has the vectors (1,2,0) and (-1,0,2) as its adjacent sides.

2(b). (8 points) Find two unit vectors each perpendicular to both of the vectors in part (a).

Soln:

2a) The area of the parallelogram is given by the magnitude of the cross product of the two vectors.

The cross product of the two vectors is given by the determinant

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 0 \\ -1 & 0 & 2 \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 0 \\ -1 & 2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 2 \\ -1 & 0 \end{vmatrix} \mathbf{k} = 4\mathbf{i} - 2\mathbf{j} + (-2)\mathbf{k}.$$

The magnitude of the cross product is
$||4\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}|| = \sqrt{4^2 + (-2)^2 + (-2)^2} = \sqrt{24} = 2\sqrt{6}$.

2b) The cross product of (1,2,0) and (-1,0,2), namely $4\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$, is perpendicular to both (1,2,0) and (-1,0,2). So to get two unit vectors perpendicular to both of them it is enough to get two unit vectors in the direction of $4\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$. This can be done by dividing by its magnitude which is $2\sqrt{6}$. So one unit vector is $\frac{4\mathbf{i}-2\mathbf{j}-2\mathbf{k}}{2\sqrt{6}}$ and the other is the negative of this, namely $-\frac{4\mathbf{i}-2\mathbf{j}-2\mathbf{k}}{2\sqrt{6}}$.

So the desired vectors are $\frac{4}{2\sqrt{6}}\mathbf{i} - \frac{2}{2\sqrt{6}}\mathbf{j} - \frac{2}{2\sqrt{6}}\mathbf{k} = \frac{2}{\sqrt{6}}\mathbf{i} - \frac{1}{\sqrt{6}}\mathbf{j} - \frac{1}{\sqrt{6}}\mathbf{k}$ and $-(\frac{2}{\sqrt{6}}\mathbf{i} - \frac{1}{\sqrt{6}}\mathbf{j} - \frac{1}{\sqrt{6}}\mathbf{k})$.

3. (12 points) Three vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are in the same plane if and only if their scalar triple product $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$ is zero. Show that the **vectors** (1,3,5), (3,5,7) and (5,7,9) are in the same plane. (8 points) Write a parametric equation for the line through the point (1,1,1) along the vector (1,3,5).

Soln: The scalar triple product is given by the determinant

$$\begin{vmatrix} 1 & 3 & 5 \\ 3 & 5 & 7 \\ 5 & 7 & 9 \end{vmatrix}.$$

Expanding along the first row, the determinant equals 1(45-49)-3(27-35)+5(21-25) = -4+24-20 = 0.

So the three vectors are in the same plane.

The parametric equation for the line through the point (1,1,1) along the vector (1,3,5) is given by $\mathbf{r} = (x, y, z) = (1, 1, 1) + t(1, 3, 5)$ which gives $x = 1 + t, y = 1 + 3t, z = 1 + 5t$.

4. (Challenge, 15 points) Let A, B, C be three non-collinear points. Describe the set of points P in 3-space such that $\vec{AP} \cdot (\vec{AB} \times \vec{AC}) = 0$.

Soln:

Based on the fact that three vectors are in the same plane only if their scalar triple product is zero, we see that $\vec{AP} \cdot (\vec{AB} \times \vec{AC}) = 0$ is same as saying that the vectors $\vec{AP}, \vec{AB}, \vec{AC}$ are in the same plane. This means that the points A, B, C, P must all be in the same plane. Since any three points can be made to lie in the same plane, the set of such P is simply the plane containing A, B and C. [Since they are non-collinear, there is only one plane passing through them].