1/20/2011 Springl 2011, Calculus III Quiz 1 Sitaraman

Instructions:
PLEASE PROVIDE STEP BY STEP EXPLANATIONS
ANSWERS WITHOUT EXPLANATION WILL ONLY GET 40 percent
Time Limit 30 minutes ; Total 50 points
Please read the questions carefully before answering
It is recommended that you try those problems you are most comfortable with, first.
Attempt as many as you can; Anything over 50 is extra credit.

1. (10 points) Say whether each statement is true or false. In each case, explain why it is true/false or give counterexample. If true it is NOT enough to give just one example.

(a) The vectors $\mathbf{i} + \mathbf{j}$ is same as the vector going from (3,7.1) to (4,8.1).

(b) $\mathbf{i}, -\mathbf{i}, \mathbf{j}, -\mathbf{j}, \mathbf{k}, -\mathbf{k}$ are all the unit vectors in 3-space.

Soln:

1a) True. The vector going from (3,7.1) to (4,8.1) is the given vector $\mathbf{i} + \mathbf{j}$ because 4-3 = 1 and 8.1-7.1= 1 are the components of this vector and $\mathbf{i} + \mathbf{j}$ as well.

1b) This is false. One can produce a unit vector in the direction of any vector $\mathbf{v}$ (thus there are infinitely many unit vectors!) by dividing by $||\mathbf{v}||$, the magnitude.

2(a). (10 points) Write the equation of sphere with center at (1,-1,0) and radius 5.

2(b). Describe the surface whose equation is $x + y = 1$.

Soln:

2a) $(x-1)^2 + (y-(-1))^2 + (z-0)^2 = 5^2$ which is same as $(x-1)^2 + (y+1)^2 + z^2 = 25$.

2b) This is a plane which is parallel to $z-$axis, intersecting the xy-plane along the line $x + y = 1$.

3. Describe the component of the force given by the vector (3,4,5) along the direction (0,1,1). What is the work done if the force moves an object 10 meters in this direction?

Soln: The unit vector along (0,1,1) is $\frac{(0,1,1)}{||(0,1,1)||} = (1/\sqrt{2})(0,1,1)$.

The magnitude of the component of the force represented by (3,4,5) along this direction is the scalar product of the force vector with the unit vector along $\mathbf{b}$. The unit vector along $\mathbf{b}$ is $\frac{\mathbf{b}}{||\mathbf{b}||} = \frac{(0,1,1)}{\sqrt{2}}$. So the desired scalar product is $(3,4,5)\cdot((1/\sqrt{2})(0,1,1)) = (3,4,5)\cdot(0,\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}) = 3(0)+4(1/\sqrt{2}))+5(1/\sqrt{2}) = 9/\sqrt{2}$. The actual component of the force along the unit vector is this magnitude times the unit vector, namely

$(9/\sqrt{2})((1/\sqrt{2})(0,1,1) = (0,4.5,4.5)$.

You can also find this using

$$\mathbf{Proj_b v} = \frac{\mathbf{v}\cdot\mathbf{b}}{\mathbf{b}\cdot\mathbf{b}}\mathbf{b} = \frac{(3,4,5)\cdot(0,1,1)}{(0,1,1)\cdot(0,1,1)}(0,1,1)$$

$$= \frac{0+4+5}{0+1+1}(0,1,1) = (9/2)(0,1,1) = (0,4.5,4.5)$$

Magnitude of this force (0,4.5,4.5) is $\sqrt{(9/2)^2+(9/2)^2} = 9/\sqrt{2}$.

The work done along this direction in moving 10 meters is given by 10 times the magnitude of the force in that direction which is $10(9/\sqrt{2}) = 90/\sqrt{2} = 63.64$.

4. (Challenge, 15 points) Use vectors to prove that a parallelogram is a rectangle if the diagonals are equal. You may use the fact that the magnitude of a vector $\mathbf{v} = ||\mathbf{v}|| = \sqrt{\mathbf{v}\cdot\mathbf{v}}$.

Soln:

Since the opposite sides of a parallelogram are equal in length and parallel, they can be represented by the same vectors. Let $\mathbf{u}$ represent one set of parallel sides and $\mathbf{v}$ represent the other pair. Then it is easy to see (draw a picture if you wish to see it more clearly) that $\mathbf{u}+\mathbf{v}$ represents one of the diagonals and $\mathbf{u}-\mathbf{v}$ represents the other.

If the diagonals are equal in length, then $||\mathbf{u}+\mathbf{v}|| = ||\mathbf{u}-\mathbf{v}||$.

Writing the magnitudes as square root of scalar product we get $\sqrt{(\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v})} = \sqrt{(\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v})}$.

Squaring both sides, we get $(\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) = (\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v})$.

expanding the scalar products, we get $\mathbf{u}\cdot\mathbf{u}+2\mathbf{u}\cdot\mathbf{v}+\mathbf{v}\cdot\mathbf{v} = \mathbf{u}\cdot\mathbf{u}-2\mathbf{u}\cdot\mathbf{v}+\mathbf{v}\cdot\mathbf{v}$. Here we used the fact that $\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u}$ in order to simplify. (continued on next page)

Canceling like terms, we get $-2\mathbf{u}\cdot\mathbf{v} = 2\mathbf{u}\cdot\mathbf{v}$ which reduces to $4\mathbf{u}\cdot\mathbf{v} = 0$. This means $\mathbf{u}\cdot\mathbf{v} = 0$ which means $\mathbf{u}$ and $\mathbf{v}$ must be perpendicular. But if the sides are perpendicular to each other and opposite sides are equal in length the parallelogram becomes a rectangle.