HOWARD UNIVERSITY MATH DEPARTMENT

5/3/2011 Spring 2011, Calculus III Final Exam

Instructions:
PLEASE PROVIDE STEP BY STEP EXPLANATIONS
Time Limit 120 minutes ; Total 200 points
Please read the questions carefully before answering
It is recommended that you try those problems you are most comfortable with, first.
**ANSWER ANY 10; EACH 20 POINTS**

1. For the points A (1,-1,2), B (2,-3,0), C (-1,-2,0), D (2,1,-1) find the volume of the parallelopiped that has the vectors $\vec{AB}, \vec{AC}$ and $\vec{AD}$ as adjacent edges.

Soln:

$$|\vec{AB}\cdot(\vec{AC}\times\vec{AD})| =< 1,-2,-2 > \cdot(< -2,-1,-2 > \times < 1,2,-3 >)$$

$$= \begin{vmatrix} 1 & -2 & -2 \\ -2 & -1 & -2 \\ 1 & 2 & -3 \end{vmatrix} = 29.$$

2. For the circular helix $x = cost,\ y = sint,\ z = t$ find the unit tangent vector $\mathbf{T}(\pi)$ at $t = \pi$. Also find the arc length of this curve from $t = 0$ to $\pi$.

Soln:

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{||\mathbf{r}'(t)||} = \frac{< -sint, cost, 1 >}{\sqrt{2}} \Rightarrow \mathbf{T}(\pi) =< 0, \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}} >$$

$$Arc\ length\ = \int_0^\pi ||\mathbf{r}'(t)||dt = \sqrt{2}\pi$$

3. The electric potential at a point is given by $V(x,y) = e^{-2x}cos2y$. Find the rate of change of potential at $(0,\pi/4)$ along the direction $\mathbf{i}+\mathbf{j}$. Find the direction along which the potential decreases most rapidly at $(0,\pi/4)$.

Soln:

Dividing $\mathbf{i}+\mathbf{j}$ by its magnitude we get a unit vector in the same direction: $\mathbf{u} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$. The rate of change of potential at $(0,\pi/4)$ along the direction $\mathbf{i}+\mathbf{j}$ is simply the directional derivative $D_{\mathbf{u}}V(0,\pi/4) = \nabla V\cdot\mathbf{u}$ at $(0,\pi/4)$.

$$\nabla V(0,\pi/4) =< -2e^{-2x}cos(2y), e^{-2x}(-2sin(2y)) > |_{(0,\pi/4)} ==< 0,-2 >= -2\mathbf{j}$$

$$D_{\mathbf{u}}V(0,\pi/4) = \nabla V\cdot\mathbf{u} = -2\mathbf{j}\cdot\frac{1}{\sqrt{2}}(\mathbf{i}+\mathbf{j}) = -\sqrt{2}.$$

The potential decreases most rapidly in the direction $-\nabla V = -(-2\mathbf{j}) = 2\mathbf{j}$.

4. Find parametric equations for the tangent line to the curve of intersection of the paraboloid $z = x^2 + y^2$ and the ellipsoid $x^2 + 4y^2 + z^2 = 9$ at the point (1,-1,2). [Hint: Use the tangent planes to the two surfaces at (1,-1,2)].

Soln :

Note: The curve of intersection is NOT given by subtracting: $3y^2 + z^2 = 9 - z \Rightarrow 3y^2 + (z + \frac{1}{2})^2 = \frac{37}{4}$. This is an ellipse that is the projection of the curve of intersection but not the actual curve. You need to use both equations to represent the cuve of intersection.

The tangent line to the curve of intersection lies on the tangent planes to both of these surfaces at this point. So it is perpendicular to the normals to both of their tangent planes. Now the normals are given by the gradients. A vector perpendicular to both of them will be given by their cross product.

Let $F = x^2 + y^2 - z$ and $G = x^2 + 4y^2 + z^2$. Then a vector parallel to the tangent line of curve of intersection is $\mathbf{v} = \nabla F \times \nabla G =< 2x, 2y, -1 > \times < 2x, 8y, 2z >$ . Now at (1,-1,2), $\nabla F \times \nabla G =< 2, -2, -1 > \times < 2, -8, 4 >$ .

The cross product is given by

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -2 & -1 \\ 2 & -8 & 4 \end{vmatrix} = -16\mathbf{i} - 10\mathbf{j} - 12\mathbf{k}.$$

Then the equation for the tangent line at (1,-1,2) is given by

$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v} =< 1, -1, 2 > +t < -16, -10, -12 >$ .

The parametric equations are: $x = 1 - 16t, y = -1 - 10t, z = 2 - 12t$.

**Alternative approach using intersection of tangent planes**

Tangent planes are $(\mathbf{r}- < 1, -1, 2 >) \cdot \nabla F = 0$ and $(\mathbf{r}- < 1, -1, 2 >) \cdot \nabla G = 0$.

You can try to solve for two of the variables in terms of the third and get the parametric equation of the line of intersection. Amounts to the same thing as using the cross product of the gradients (normals to the tangent planes).

5. Find absolute maximum and minimum of the following:

$f(x, y) = x^2 + 2y^2 - x$ on the disk $x^2 + y^2 \leq 4$.

Soln:

First we find the critical points:

$f_x = 0$ and $f_y = 0$ gives $2x - 1 = 0, 4y = 0$. Solving these two equations we get $x = 1/2, y = 0$. This point is inside the given disk.

On the boundary circle $x^2+y^2 = 4$ we have $f(x, y) = x^2+2y^2-x = x^2+2(4-x^2)-x = 8 - x - x^2$. This is a function of one variable and $f'(x) = -1 - 2x = 0$ when $x = -1/2$. When $x = -1/2$ we have $y = \pm\sqrt{4 - x^2} = \pm\sqrt{15/4} = \pm\sqrt{15}/2$.

Compare values of $f(x, y)$ :

$f(1/2, 0) = -1/4; f(-1/2, \pm\sqrt{15}/2) = 33/4$.

So absolute max is 33/4 and absolute min is -1/4.

6. If $z = f(x, y)$ then use the conversion equations $x = rcos\theta,\ y = rsin\theta$ and the chain rule to write $\frac{\partial z}{\partial r}$ and $\frac{\partial z}{\partial \theta}$ each as a combination of $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$. Then show that

$$\left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2$$

Soln: Plug in these into the left hand side:

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} cos\theta + \frac{\partial z}{\partial y} sin\theta, \quad \frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}(-rsin\theta) + \frac{\partial z}{\partial y}(rcos\theta)$$

7. Identify the surface $z = y^2 - x^2$. Show that it has one critical point and that point is a saddle point.

Soln: The curve is a hyperbolic paraboloid.

$z_x = -2x = 0, z_y = 2y = 0$ gives $x = 0, y = 0$. The hessian $D = z_{xx}z_{yy} - z_{xy}^2 = -4$ is negative, so it is a saddle point at (0,0).

8. Evaluate the double integral $\int\int ydA$ over the region R in the first quadrant enclosed between the circle $x^2 + y^2 = 25$ and the line $x + y = 5$.

Soln:

$$\int_0^5 dx \int_{5-x}^{\sqrt{25-x^2}} ydy = \frac{125}{6}$$

9. Evaluate the iterated integral by converting to polar coordinates:

$$\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2) dydx$$

Soln:

$$\int_0^{\pi/2} \int_0^1 r^2(rdr)d\theta = \frac{\pi}{8}$$

10. Find the surface area of the portion of the paraboloid $x = ucosv, y = usinv, z = u^2$ where $1 \le u \le 2, 0 \le v \le 2\pi$.

Soln:

$$||\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}|| = || < cosv, sinv, 2u > \times < -usinv, ucosv, 0 > ||$$

$$= || < -2u^2cosv, -2u^2sinv, u > || = \sqrt{4u^4 + u^2} = u\sqrt{2u^2 + 1}.$$

So the surface area is

$$\int_0^{2\pi} dv \int_1^2 u\sqrt{4u^2 + 1}\ du = (17^{3/2} - 5^{3/2})\pi/6$$

11. Compute the divergence, curl and the divergence curl of the vector field

$$\mathbf{F}(x, y, z) = x^2\mathbf{i} - 2\mathbf{j} + yz\mathbf{k}.$$

Soln:

$$div\mathbf{F} = 2x + y, \quad curl\mathbf{F} = z\mathbf{i}, \quad div(curl\mathbf{F}) = 0$$

12. State Green's theorem. Evaluate the line integral $\oint ydx + xdy$ over the positively oriented unit circle, by using Green's theorem.

Soln:

$$\oint ydx + xdy = \int\int \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA = \int\int (0)dA = 0.$$