HOWARD UNIVERSITY MATH DEPARTMENT

4/27/2011 Spring 2011, Calculus III Prospective Grads Final Exam

Instructions:
PLEASE PROVIDE STEP BY STEP EXPLANATIONS
Time Limit 120 minutes ; Total 200 points
Please read the questions carefully before answering
It is recommended that you try those problems you are most comfortable with, first.
ANSWER ANY 10; EACH 20 POINTS

1. For the points A (1,0,1), B (0,2,3), C (2,1,0) find the area of the triangle ABC using cross-product of vectors.

Soln: This is problem 15 on page 839. Solve it by taking the magnitude of $\vec{AB} \times \vec{AC}$. You should get $\sqrt{26}/2$.

2.Confirm that s is an arc-length parametrization by checking that $||\frac{d\mathbf{r}}{ds}|| = 1$ and then use it to find the curvature $\kappa(s)$ of the curve:

$$\mathbf{r}(s) = sin(1+\frac{s}{2})\mathbf{i} + cos(1+\frac{s}{2})\mathbf{j} + \sqrt{3}(1+\frac{s}{2})\mathbf{k}$$

Soln:
12.5-17
Answer: $\kappa = |\mathbf{r}''(s)| = 1/4$.

3. Find the directional derivative of $f(x,y) = 4x^3y^2$ at (2,1) in the direction of the vector $\mathbf{a} = 4\mathbf{i} - 3\mathbf{j}$.

Soln: 13.6-9
Dividing $4\mathbf{i} - 3\mathbf{j}$ by its magnitude =5 we get a unit vector in the same direction: $\mathbf{u} = \frac{4}{5}\mathbf{i} - \frac{3}{5}\mathbf{j}$.
The directional derivative $D_{\mathbf{u}}f(2,1) = \nabla f \cdot \mathbf{u}$.
Answer is 0.

4. Find the equation for the tangent plane and the parametric equations for the normal line to the surface $z = xe^{-y}$ at the point (1,0,1).

Soln : 13.7-7
The tangent plane is given by using the normal vector which is obtained using the gradient at the given point. Normal line is also got by using normal vector.
Answer: $x - y - z = 0$ and $x = 1 + t, y = -t, z = 1 - t$.

5. Find absolute maximum and minimum of the following:
$f(x,y) = xy - x - 3y$ on the triangle with vertices (0,0), (0,4) and (5,0).

Soln: 13.8-31
First find the critical points:
$f_x = 0$ and $f_y = 0$ gives $x = 3, y = 1$. At this point $f(3,1) = 3 - 3 - 3 = -3$. This point is inside the given triangle.

On the line joining (5,0) and (0,4) given by $4x + 5y = 20$ we have $f(x,y) = x(4 - \frac{4}{5}x) - x - 3(4 - \frac{4}{5}x)$. This is a function of x only. On simplifying and setting $f'(x) = 0$ we get $x = 27/8, y = 13/10$. Plugging this back into f we get $f = -2.9$ approximately.

On the boundary triangle when $x = 0$ you get $f = -3y$. This single variable function of y has no critical points. Similarly on the other lines of the triangle $y = 0$ there are no critical points.

The only points left to consider are the "boundaries of the boundaries"–the endpoints of the three sides of the triangle, namely (0,0), (0,4) and (5,0).

At these points $f = 0, -12$ and -5 respectively.

Among all the values of f discussed so far, namely -3,-2.9,0,-12 and -5 we have absolute max = 0 and absolute min = -12.

6. Use an appropriate form of the chain rule to find $\frac{dw}{dt}$ given that $w = 5x^2y^3z^4,\ x = t^2,\ y = t^3, z = t^5$.

Soln: 13.5-7

Use formula (6) on page 950. Answer $165t^{32}$.

7. Identify the surface $z = 1 - x^2 - y^2$. Identify all critical points and find the local maxima and minima. Then find the absolute maxima and minima by inspecting the graph of the surface.

Soln:13.8-1b

The curve is a paraboloid facing down with vertex at (0,0,1).

$z_x = -2x = 0, z_y = -2y = 0$ gives $x = 0, y = 0$. So (0,0) is the lone critical point. The hessian $D = z_{xx}z_{yy} - z_{xy}^2$ is 0, so we cannot conclude if this is a maximum or minimum. But from the graph it is easy to see that (0,0,1) is the absolute maximum.

8. Evaluate the double integral $\int\int 4xy^3 dA$ over the region R in the $xy-$plane described by the inequalities $-1 \leq x \leq 1, -2 \leq y \leq 2$.

Soln: 14.1-13

$$\int_{-1}^{1} dx \int_{-2}^{2} 4xy^3 dy = 0$$

9. Use a double integral in polar coordinates to find the area enclosed by the cardioid $r = 1 - cos\theta$.

Soln: 14.3-7

$$Area = \int_{0}^{2\pi} \int_{0}^{1-cos\theta} (rdr)d\theta = 3\pi/2$$

Use $cos^2\theta = (1 + cos2\theta)/2$ to integrate $cos^2\theta$.

10. Find the volume of the solid in the first octant bounded by the three coordinate planes and the plane $3x + 6y + 4z = 12$.

Soln: 14.5-15

This can be done using a double or a triple integral. In both cases start by looking at the region in the $xy-$plane under the surface. Here it will be $3x + 6y = 12$ by setting $z = 0$. This line intersects the axes at $x = 4$ and $y = 2$. So you can start by letting x go from 0 to 4 or y go from 0 to 2. In the first case for each x the y will go from 0 $to(12 - 3x)/6 = 2 - \frac{x}{2}$.

If it is triple integral z will go from 0 to $z = (12 - 3x - 6y)/4$ and the integrand will be $dzdydx$.

If it is double integral the integrand will be $(12 - 3x - 6y)/4$ $dydx$.

11. Compute the divergence, curl and the divergence curl of the vector field

$$\mathbf{F}(x, y, z) = 7y^3z^2\mathbf{i} - 8x^2z^5\mathbf{j} - 3xy^4\mathbf{k}.$$

Soln: 15.1-19 Answer: divergence is 0, divergence curl is always 0. curl is obtained using the cross-product like determinant. curl is $(40x^2z^4 - 12xy^3)\mathbf{i} + (14y^3z + 3y^4)\mathbf{j} + (16xz^5 + 21y^2z^2)\mathbf{k}$.

12. State Green's theorem. Evaluate the line integral $\oint y^2dx + x^2dy$ where C is the square with vertices (0,0),(1,0),(1,1) and (0,1) by using Green's theorem.

Soln: 15.4-1

$$\oint y^2dx + x^2dy = \int\int \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA = \int\int (2x - 2y)dA = 0.$$