Spring 05 Test 3 Solutions Calculus III 158-01 Sitaraman 3/28/05

1. Find the vector equation of the line tangent to the graph of $\mathbf{r}(t) = sin(t)\mathbf{i} + cos(t)\mathbf{j} + t\mathbf{k}$ at $t = 0$.

Soln: A vector parallel to the tangent line is given by letting $t = 0$ in $\mathbf{r}'(t) = cos(t)\mathbf{i} - sin(t)\mathbf{j} + \mathbf{k}$. We get $\mathbf{r}'(0) = \mathbf{i} + \mathbf{k}$. A point on the curve is given by $\mathbf{r}(0) = \mathbf{j}$. So the vector equation of the tangent line is given by $\mathbf{r}(t) = \mathbf{j} + t(\mathbf{i} + \mathbf{k})$.

2. A bug is moving along a curve defined by $\mathbf{r}(t) = 2e^{-t}\mathbf{i} + e^{-t}\mathbf{j} + 2e^{-t}\mathbf{k}$. Find the distance travelled by the bug from $t = 0$ to $t = 1$.

Soln: The distance is given by the arc length from $t = 0$ to $t = 1$. This is given by

$$\int_0^1 \sqrt{(x'(t)^2 + y'(t)^2 + z'(t)^2}\ dt$$

$$= \int_0^1 \sqrt{(-2e^{-t})^2 + (-e^{-t})^2 + (-2e^{-t})^2}\ dt$$

$$= \int_0^1 \sqrt{9}e^{-t}dt = -3(e^{-1} - 1) = 1.89636.$$

So the distance is 1.89636.

3. In problem 2, find the bug's velocity and speed at time t. Is there ever a moment when the bug is going in the vertical direction? (i.e, along $\mathbf{k}$).

Soln: Velocity is given by $\mathbf{r}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k} = (-e^{-t})(2\mathbf{i} + \mathbf{j} + 2\mathbf{k})$. Speed is given by $\frac{ds}{dt} = |\mathbf{r}'(t)| = \sqrt{(x'(t)^2 + y'(t)^2 + z'(t)^2} = 3e^{-t}$. If it is going in vertical direction, then velocity vector will be given by $\mathbf{r}'(t) = a\mathbf{k}$ where a is a scalar quantity. But from the equation for $\mathbf{r}'(t)$ we see that it will never be of this form because all the three components are always non-zero. So it is never going in just the vertical direction.

4. Find the curvature of the bug's path at time 0 and the normal and tangential components of its acceleration at time t.

Soln: The tangential component $a_T = \frac{d^2s}{dt^2}$ which is $\frac{d}{dt}(3e^{-t}) = -3e^{-t}$. The normal component is given by $\frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|} = |\mathbf{v} \times \mathbf{a}|/(\frac{ds}{dt}) =$

$|(-e^{-t})(2\mathbf{i}+\mathbf{j}+2\mathbf{k}) \times (e^{-t}(2\mathbf{i}+\mathbf{j}+2\mathbf{k}))|/(\frac{ds}{dt})$. But the numerator is zero because velocity and acceleration are both multiples of the same vector $2\mathbf{i}+\mathbf{j}+2\mathbf{k}$ and hence parallel and hence their cross product is zero. Hence curvature is also zero because curvature is given by $a_N/(\frac{ds}{dt})^2$.

5. Describe the level surfaces of $f(x, y, z) = xyz$ in words and using graph.

Soln: The level survaces of this have cross sections that are hyperbolas given by $xy = k/z$ as z varies.

6. (bonus problem) Prove that, for any particle moving with a velocity $\mathbf{v}(t)$, acceleration $\mathbf{a}(t)$ and speed $v(t)$, $\mathbf{v} \cdot \mathbf{a} = \frac{1}{2}(\frac{d(v(t)^2)}{dt})$.

Soln:

Method A: $\mathbf{v}(t) = v(t)\mathbf{T}(t)$ where T is the unit tangent vector. So $\mathbf{v} \cdot \mathbf{a} = v(t)\mathbf{T} \cdot (a_T\mathbf{T} + a_N\mathbf{N}) = v(t)a_T = v(t)v'(t) = (1/2)\frac{d(v(t)^2)}{dt}$. [Here $\mathbf{T} \cdot \mathbf{N} = 0$ because they are perpendicular to each other].

Method B: $\mathbf{v} \cdot \mathbf{a} = x'(t)x"(t) + y'(t)y"(t) + z'(t)z"(t)$
$= (1/2)\frac{d}{dt}(x'(t)^2 + y'(t)^2 + z'(t)^2)$.