Howard University Math Department

Instructions:

PLEASE PROVIDE STEP BY STEP EXPLANATIONS

WRITING ONLY ANSWERS WILL NOT GET FULL CREDIT

Time Limit 30 minutes

Please read the questions carefully before answering

1. (15 points) Use strong induction to prove that postage of 8 cents or more can be obtained by using only 3 cent and 5 cent stamps.

   Base case: Enough to check for 8, 9 and 10. Reason will become clear below. $8 = 5+3$ and $9 = 3 + 3 + 3$ and $10 = 5 + 5$. In general for postage stamp problems you check the same number of cases as the smaller number, in this case 3.

   Assumption: Assume true for all k such that $7 < k < n$. In other words, any such k can be broken down into sums of 3's and / or 5's.

   Proving for n using numbers smaller than n : We can go from $n-3$ to n. Since we already proved for 8, 9 and 10, enough to start with $n = 11$.

   $n-3$ satisfies the condition that it lies between 7 and n and so the assumption applies for $n - 3$.

   So $n - 3$ is a sum of 3's and / or 5's, say $n - 3 = 3x + 5y$.

   But then $n = (n - 3) + 3$ so it should also be a sum of 3's and / or 5's, namely $n - 3 + 3 = 3x + 5y + 3 = 3(x + 1) + 5y$.

   This concludes the proof.

2. (15 points) Show using strong induction that for any natural number $n \geq 8$, the Fibonacci numbers defined by $F_1 = 1, F_2 = 1, F_3 = 2, F_4 = 3, F_5 = 5, F_6 = 8, ..., F_n = F_{n-1} + F_{n-2}$ satisfy

   $$F_n < \left(\frac{5}{3}\right)^{n-2}.$$

   Note that it *doesn't* work for n up to 7. For example, $(5/3)^{7-2} = 12.86$ but $F_7 = 13$.

   Solution:

   Check this is true for $n = 8$ and 9.

   This is because we will need *two previous steps* to prove the n-th step.

Since by strong induction we assume the statement is true for all $k < n$, it could certainly be assumed true for $n-1$ and $n-2$.

So we will assume that $F_{n-1} < (5/3)^{n-3}$ and $F_{n-2} < (5/3)^{n-4}$ and try to prove it for n.

Combining the two inequalities we get

$$F_n = F_{n-1} + F_{n-2} < \left(\frac{5}{3}\right)^{n-3} + \left(\frac{5}{3}\right)^{n-4} = \left(\frac{5}{3}\right)^{n-4}\left(\frac{5}{3}+1\right)$$

Now $(5/3)+1 = 8/3$ and $(8/3) < (5/3)^2 = 25/9$ because $8/3 = 24/9$ (multiply above and below by 3).

Therefore in the last inequality would still hold if we replace $8/3$ by $(5/3)^2$.

We get

$$F_n < \left(\frac{5}{3}\right)^{n-4}\left(\frac{5}{3}+1\right) < \left(\frac{5}{3}\right)^{n-4}\left(\frac{5}{3}\right)^{2} = \left(\frac{5}{3}\right)^{n-2}$$

Thus we have proved the statement for n and the proof is complete.

3. (Extra credit 15 points) Prove : For any natural number $n \geq 4$ you can find positive integers x, y such that $n = 2x + 5y$.

   This is basically the problem of making every postage starting with 4 cents using 2 cent and 5 cent stamps.

   It says you can add multiples of 2 and 5 to get any number starting with 4. Note that it is not possible for 3.