Howard University Math Department

PLEASE PROVIDE STEP BY STEP SOLUTIONS. TOTAL 100 POINTS. TIME 50 MINUTES.

1. (16 points) Let a, b, c be positive integers. Show that $gcd(a, b, c) = gcd(gcd(a, b), c)$. Use this to prove that $gcd(a, b, c) = ax + by + cz$ for some integers x, y, z.
2. (16 points) Can you prove that there are infinitely many primes of the form $n^2 + 1$ in the way we proved that there are infinitely many primes of the form $3n + 2$? Say, by assuming N is the product of the finitely many primes of the form $n^2 + 1$ and then looking at $N^2 + 1$?
3. (16 points) Show that for any integer $n > 1$, $\dfrac{n^k - 1}{n - 1}$ is prime means k is prime.
4. (16 points) True or false? Prove if true and give counterexample if not true.

   (a) Given two integers m, n any integer N can be represented as $N = mx + ny$ for some integers x, y.

   (b) $gcd(m, n)$ is the smallest positive integer that can be written in the form $mx + ny$ for some integers x, y.
5. (16 points) (a) Find all possible pythagorean triples (not necessarily primitive) that are consecutive integers.

   (b) What about Pythagorean triples of form $(n, n + 1, n + 3)$?
6. (20 points) By drawing lines from $(1, 1)$ parametrize all *rational points* on $x^2 + y^2 = 2$.
7. (Challenge: 20 points) Show that the Pythagorean triples with $b = a+1$ (i.e, first two numbers being consecutive) are in 1-1 correspondence with the solutions of $x^2 - 2y^2 = -1$.