Howard University Math Department

SOLUTION FOR PROBLEM 16

Question 16: Can a primitive pythagorean triple have two primes in it? How often? How often do they have one prime ? How about all three being prime?

Solution:
First, it is easy to see that no three primes $p < q < r$ cannot be a Pythagorean triple:
First note that any 3 primes are automatically a primitive triple.
The only possibility for p or q to be st would be if $s = 1$ or $t = 1$.
Let us say $t = 1$ and $s = p$ is a prime. (Other cases are similar).
Then using the standard parametrization $q = (p^2 - 1)/2, r = (p^2 + 1)/2$.
If $p = 2$ then $q = 3/2$ so that is not possible.
If $p = 3$ then $q = 4$ so that is also not possible.
If $p > 3$ then $(p - 1)/2 > 1$ and $q = (p^2 - 1)/2 = (p - 1)(p + 1)/2$ cannot be a prime.

CASE WHEN ONE OF a, b IS A PRIME.

Suppose WLOG a is a prime. (The other case is similar).
If $a = 2$ then we have $4 + b^2 = c^2$. Even if $c = b + 1$ we need $4 = 2b + 1$ which will be impossible because $b > a \implies 2b+1 > 4$. (The cases $b = 1$ or $b = 2$ do not give Pythagorean triples).
So let $a > 2$. Then a is an odd prime.

Claim: $a^2 = (c + b)(c - b) \implies c + b = a^2, c - b = 1$
Proof: $c + b, c - b$ can have only 2 in common and a^2 is odd. So a^2 has to divide one of them completely (assuming unique factorization theorem). The other has to be 1.

$$c + b = a^2, c - b = 1 \implies 2b + 1 = a^2 \implies b = (a^2 - 1)/2.$$

Since $a > 2, (a^2 - 1)/2$ is prime (possibly) only if $a = 3$. But then $b = 4, c = 5$ so we get only a and c being prime. [If $a > 3$ then $(a - 1)/2 > 1$ and $b = (a^2 - 1)/2 = (a - 1)(a + 1)/2$ cannot be a prime]. But it is still possible c is prime and that means $(a^2 + 1)/2 = c$ is prime.

So in conclusion we get that **if one of a or b is prime then the other is not, and we get c to be prime only if** $c = (a^2 + 1)/2$ (or $= (b^2 + 1)/2$) **is prime.**

Examples:
First note that, we can see from the above that in this case also we get a primitive triple.
3 is prime and $(3^2 + 1)/2 = 5$ is a prime. They are part of $(3, 4, 5)$.
5 is prime and $(5^2 + 1)/2 = 13$ is a prime. They are part of $(5, 12, 13)$.
7 is prime but $(7^2 + 1)/2 = 25$ is not a prime. No Pythagorean triple.
11 is prime and $(11^2 + 1)/2 = 61$ is a prime. They are part of $(5, 60, 61)$.
13 is prime but $(13^2 + 1)/2 = 85$ is not a prime. No Pythagorean triple.
17 is prime but $(17^2 + 1)/2 = 145$ is not a prime. No Pythagorean triple.
19 is prime and $(19^2 + 1)/2 = 181$ is a prime. They are part of $(19, 180, 181)$.