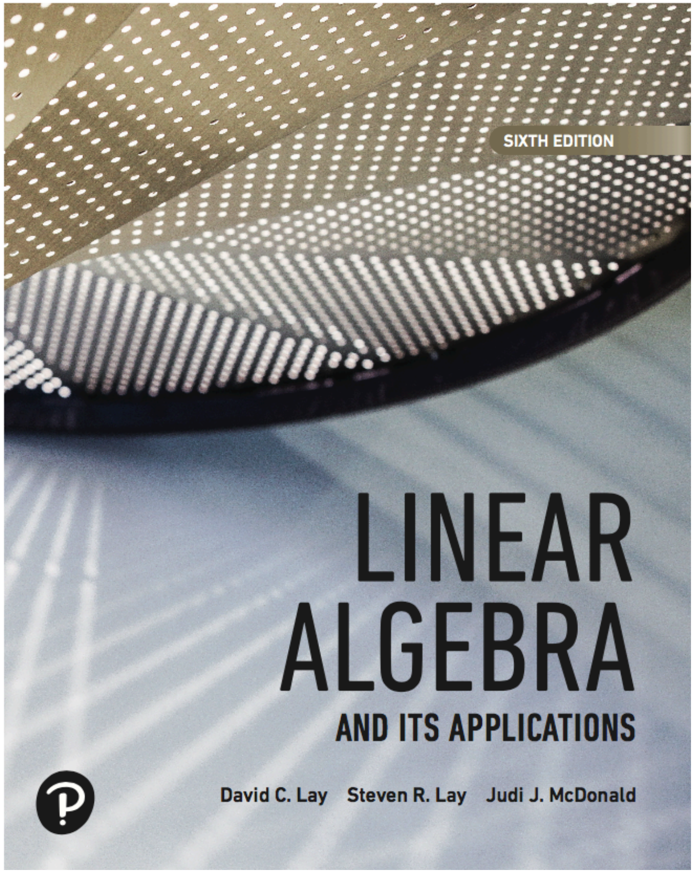
1 Linear Equations in Linear Algebra

1.3

VECTOR EQUATIONS



VECTOR EQUATIONS

Vectors in $\mathbb{R}^2$

- A matrix with only one column is called a **column vector**, or simply a **vector**.

- An example of a vector with two entries is

$$\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix},$$

where w_1 and w_2 are any real numbers.

- The set of all vectors with 2 entries is denoted by $\mathbb{R}^2$ (read "r-two").

PURPOSE OF VECTOR EQUATIONS

Vector equations are a way to write linear systems using matrices and vectors.

They help us to see more patterns among these equations by making the whole system look like a simple equation such as 2x = 3.

- An example of a vector equation:

$$A\,\mathbf{w} = \mathbf{v}\text{ , where } A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix},\ \mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix},\ \mathbf{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

This is same as $w_1 + 2w_2 = 3,\ 3w_1 + 4w_2 = 4$

VECTOR EQUATIONS

- The $\mathbb{R}$ stands for the real numbers that appear as entries in the vector, and the exponent 2 indicates that each vector contains 2 entries.
- Two vectors in $\mathbb{R}^2$ are **equal** if and only if their corresponding entries are equal.
- Given two vectors **u** and **v** in $\mathbb{R}^2$, their **sum** is the
- vector $\mathbf{u}+\mathbf{v}$ obtained by adding corresponding entries of **u** and **v**.
- Given a vector **u** and a real number *c*, the **scalar multiple** of **u** by *c* is the vector $c\mathbf{u}$ obtained by multiplying each entry in **u** by *c*.

VECTOR EQUATIONS

- **Example 1:** Given $\mathbf{u} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$, find $4\mathbf{u}$, $(-3)\mathbf{v}$ and $4\mathbf{u} + (-3)\mathbf{v}$.

Solution: $4\mathbf{u} = \begin{bmatrix} 4 \\ -8 \end{bmatrix}$, $(-3)\mathbf{v} = \begin{bmatrix} -6 \\ 15 \end{bmatrix}$ and

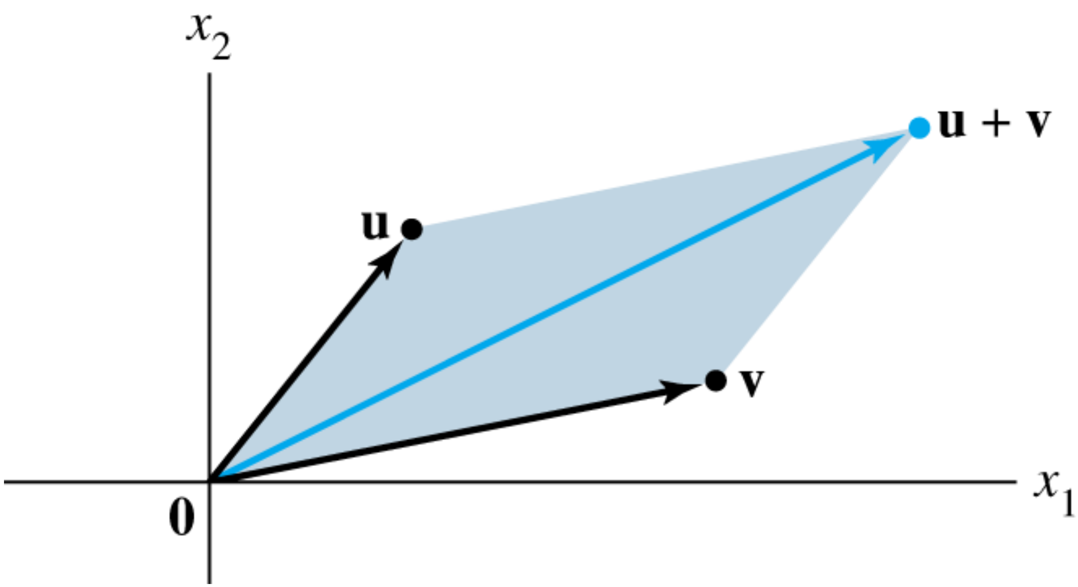
$$4\mathbf{u} + (-3)\mathbf{v} = \begin{bmatrix} 4 \\ -8 \end{bmatrix} + \begin{bmatrix} -6 \\ 15 \end{bmatrix} = \begin{bmatrix} -2 \\ 7 \end{bmatrix}$$

GEOMETRIC DESCRIPTIONS OF $\mathbb{R}^2$

- Consider a rectangular coordinate system in the plane. Because each point in the plane is determined by an ordered pair of numbers, *we can identify a geometric point (a, b) with the column vector* $\begin{bmatrix} a \\ b \end{bmatrix}$.
- So we may regard $\mathbb{R}^2$ as the set of all points in the plane.

PARALLELOGRAM RULE FOR ADDITION

- If **u** and **v** in $\mathbb{R}^2$ are represented as points in the plane, then $\mathbf{u}+\mathbf{v}$ corresponds to the fourth vertex of the parallelogram whose other vertices are **u**, **0**, and **v**. See the figure below.



VECTORS IN $\mathbb{R}^3$ and $\mathbb{R}^n$

- Vectors in $\mathbb{R}^3$ are 3×1 column matrices with three entries.
- They are represented geometrically by points in a three-dimensional coordinate space, with arrows from the origin.
- If n is a positive integer, $\mathbb{R}^n$ (read "r-n") denotes the collection of all lists (or *ordered n-tuples*) of n real numbers, usually written as $n\times 1$ column matrices, such as

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}.$$

ALGEBRAIC PROPERTIES OF $\mathbb{R}^n$

- The vector whose entries are all zero is called the **zero vector** and is denoted by **0**.
- For all **u**, **v**, **w** in $\mathbb{R}^n$ and all scalars c and d:

(i) $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$

(ii) $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$

(iii) $\mathbf{u}+\mathbf{0}=\mathbf{0}+\mathbf{u}=\mathbf{u}$

(iv) $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$

where $-\mathbf{u}$ denotes $(-1)\mathbf{u}$

(v) $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$

(vi) $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$

LINEAR COMBINATIONS

(vii) $c(d\mathbf{u}) = (cd)(\mathbf{u})$

(viii) $1\mathbf{u} = \mathbf{u}$

- Given vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ in $\mathbb{R}^n$ and given scalars $c_1, c_2, \ldots, c_p$, the vector $\mathbf{y}$ defined by

$$\mathbf{y} = c_1\mathbf{v}_1 + \ldots + c_p\mathbf{v}_p$$

is called a **linear combination** of $\mathbf{v}_1, \ldots, \mathbf{v}_p$ with **weights** $c_1, \ldots, c_p$.

- The weights in a linear combination can be any real numbers, including zero.

LINEAR COMBINATIONS

- **Example 2:** Let $\mathbf{a}_1 = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$.

Determine whether **b** can be generated (or written) as a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$. That is, determine whether weights x_1 and x_2 exist such that

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 = \mathrm{b} \quad \text{----}(1)$$

If vector equation (1) has a solution, find it.

Solution: Use the definitions of scalar multiplication and vector addition to rewrite the vector equation

$$x_1\begin{bmatrix}1\\-2\\-5\end{bmatrix}+x_2\begin{bmatrix}2\\5\\6\end{bmatrix}=\begin{bmatrix}7\\4\\-3\end{bmatrix},$$

$\uparrow \mathbf{a}_1$ $\uparrow \mathbf{a}_2$ $\uparrow \mathbf{b}$

which is same as

$$\begin{bmatrix}x_1\\-2x_1\\-5x_1\end{bmatrix}+\begin{bmatrix}2x_2\\5x_2\\6x_2\end{bmatrix}=\begin{bmatrix}7\\4\\-3\end{bmatrix}.$$

LINEAR COMBINATIONS

and $$\begin{bmatrix} x_1 + 2x_2 \\ -2x_1 + 5x_2 \\ -5x_1 + 6x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}. \qquad \text{----(2)}$$

- The vectors on the left and right sides of (2) are equal if and only if their corresponding entries are both equal. That is, x_1 and x_2 make the vector equation (1) true if and only if x_1 and x_2 satisfy the following system.

$$\begin{aligned} x_1 + 2x_2 &= 7 \\ -2x_1 + 5x_2 &= 4 \\ -5x_1 + 6x_2 &= -3 \end{aligned} \qquad \text{----(3)}$$

LINEAR COMBINATIONS

- To solve this system, row reduce the augmented matrix of the system as follows.

$$\begin{bmatrix} 1 & 2 & 7 \\ -2 & 5 & 4 \\ -5 & 6 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 7 \\ 0 & 9 & 18 \\ 0 & 16 & 32 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 7 \\ 0 & 1 & 2 \\ 0 & 16 & 32 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

- The solution of (3) is $x_1 = 3$ and $x_2 = 2$. Hence **b** is a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$, with weights $x_1 = 3$ and $x_2 = 2$. That is,

$$3\begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}.$$

LINEAR COMBINATIONS

- Now, observe that the original vectors $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{b}$ are the columns of the augmented matrix that we row reduced:

$$\begin{bmatrix} 1 & 2 & 7 \\ -2 & 5 & 4 \\ -5 & 6 & -3 \end{bmatrix}$$

$$\begin{matrix} \uparrow & \uparrow & \uparrow \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{b} \end{matrix}$$

- Write this matrix in a way that identifies its columns.

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{b} \end{bmatrix} \quad \text{----(4)}$$

LINEAR COMBINATIONS

- A vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n = \mathbf{b}$$

has the same solution set as the linear system whose augmented matrix is

$$\begin{bmatrix}\mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n & \mathbf{b}\end{bmatrix} \qquad \text{----(5)}$$

- In particular, **b** can be generated by a linear combination of $\mathbf{a}_1, \ldots, \mathbf{a}_n$ if and only if there exists a solution to the linear system corresponding to the matrix (5).

LINEAR COMBINATIONS AND SOLUTIONS

- We will see later that two vectors in $\mathbb{R}^2$ that are not parallel will generate all vectors in $\mathbb{R}^2$.
- The pair of equations ax+by = e and cx+dy = f can be translated as $x\begin{bmatrix} a \\ c \end{bmatrix} + y\begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}$
- But we know $\begin{bmatrix} e \\ f \end{bmatrix}$ is a linear combination of $\begin{bmatrix} a \\ c \end{bmatrix}$ and $\begin{bmatrix} b \\ d \end{bmatrix}$ iff the two vectors are parallel.

LINEAR COMBINATIONS AND SOLUTIONS (contd)

- $\begin{bmatrix} a \\ c \end{bmatrix}$ and $\begin{bmatrix} b \\ d \end{bmatrix}$ are parallel if one is k times another, where k is a real number.

- This means a = kb, c = bd
- This means b/a = k = d/c
- This means the two lines ax+by = e and cx+dy = f have same slope, hence parallel!

LINEAR COMBINATIONS

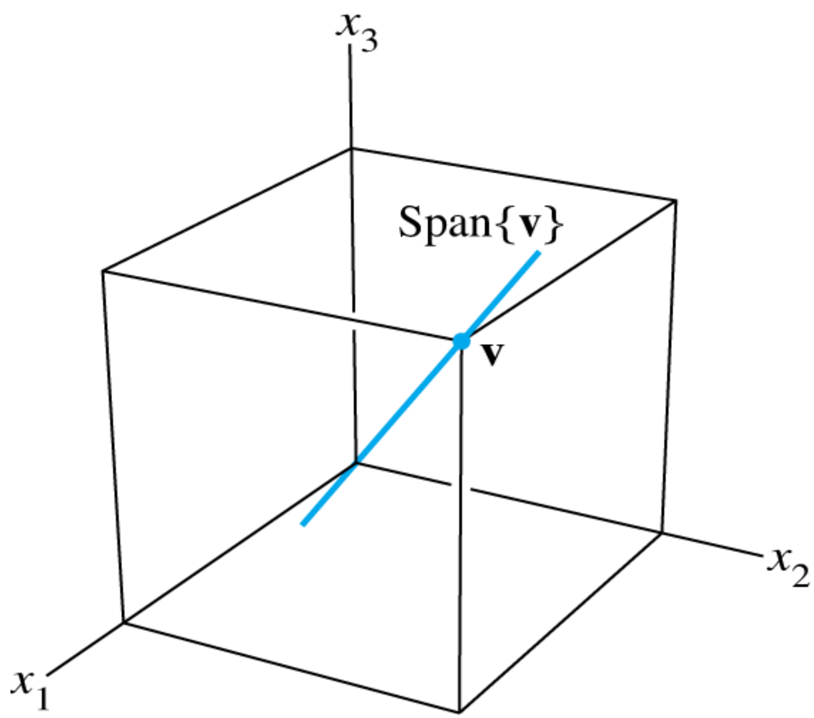
- **Definition:** If $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are in $\mathbb{R}^n$, then the set of all linear combinations of $\mathbf{v}_1, \ldots, \mathbf{v}_p$ is denoted by Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ and is called the **subset of** $\mathbb{R}^n$ **spanned** (or **generated**) **by** $\mathbf{v}_1, \ldots, \mathbf{v}_p$. That is, Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is the collection of all vectors that can be written in the form

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_p\mathbf{v}_p$$

with $c_1, \ldots, c_p$ scalars.

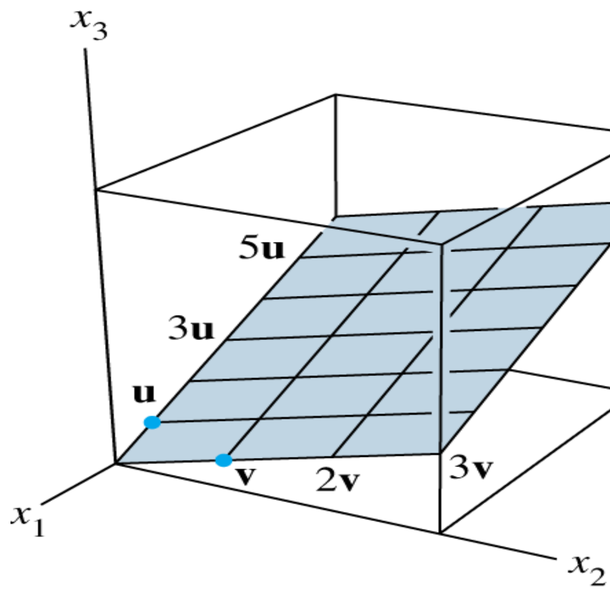
A GEOMETRIC DESCRIPTION OF SPAN {V}

- Let **v** be a nonzero vector in $\mathbb{R}^3$. Then Span {**v**} is the set of all scalar multiples of **v**, which is the set of points on the line in $\mathbb{R}^3$ through **v** and **0**. See the figure below.



A GEOMETRIC DESCRIPTION OF SPAN {U, V}

- If **u** and **v** are nonzero vectors in $\mathbb{R}^3$, with **v** not a multiple of **u**, then Span {**u, v**} is the plane in $\mathbb{R}^3$ that contains **u**, **v**, and **0**.
- In particular, Span {**u**, **v**} contains the line in $\mathbb{R}^3$ through **u** and **0** and the line through **v** and **0**. See the figure below.



Examples of Span

- The $Span\ of \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \left\{ k \begin{bmatrix} 2 \\ 3 \end{bmatrix}, k \in \mathbb{R} \right\}$.
- If $\begin{bmatrix} x \\ y \end{bmatrix} \in Span,\ then\ \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2k \\ 3k \end{bmatrix}$.
- From this you get
  $x = 2k, y = 3k \implies y = (3/2)x$.

Examples of Span

- What is the span of $\begin{bmatrix}1\\0\\1\end{bmatrix}$ *and* $\begin{bmatrix}0\\1\\1\end{bmatrix}$?
- If $\begin{bmatrix}x\\y\\z\end{bmatrix} \in Span, \begin{bmatrix}x\\y\\z\end{bmatrix} = a\begin{bmatrix}1\\0\\1\end{bmatrix} + b\begin{bmatrix}0\\1\\1\end{bmatrix}$.
- We get $x = a, y = b, z = a + b$.
- From this we get $z = x + y$ and this is the equation of the plane that is the span of the given vectors.

When do given vectors in $\mathbb{R}^m$ span all of $\mathbb{R}^m$?

- FACT :

- Let A be an $m \times n$ matrix, $\mathbf{x} \in \mathbb{R}^n$, and $\mathbf{b} \in \mathbb{R}^m$. Let $\mathbf{C}_1, \mathbf{C}_2, \ldots, \mathbf{C}_n$ be the column vectors of A.
- Then the system of equations given by $A\mathbf{x} = \mathbf{b}$ has solutions iff the vector equation $x_1\mathbf{C}_1 + x_2\mathbf{C}_2 + \ldots + x_n\mathbf{C}_n = \mathbf{b}$ has solutions iff the *matrix A has pivots in each row*.