1. (20 points) Say whether the following are true or false. If true prove it. If false give a counter-example.

(a) If A is an $n \times n$ matrix, then $det(2A) = 2^n det A$.

(b) The set of polynomials of degree exactly 1 (i.e, the linear polynomials) form a subspace of $\mathbb{P}_1$, the set all polynomials of degree at most 1.

Solution

1a. True. You can write $2A = (2I)A$ where I is the $n \times n$ identity matrix. Now $2I$ is just the diagonal matrix with 2's replacing the 1's in the diagonal. So its determinant is just the product of the diagonal entries, and equals 2^n. Then $det(2A) = det((2I)A) = det(2I)det(A) = 2^n det(A)$.

1b. False. This zero vector which is just the number zero is not a linear polynomial. Also, sum of two linear polynomials need not be a linear polynomial: $(2x + 3) + (-2x + 1) = 4$, which is not a linear.

2. (20 points) Find the null space, column space (image space), rank and nullity of the linear map $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$T(x, y, z) = (x + y, y + z, x + z)$$

Solution

The matrix of the transformation is $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

After row reduction you get the echelon matrix $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$.

This has three pivots so the columns are linearly independent and it is a one to one map and only the zero vector goes to the zero vector.

So null space is just the zero vector space and its dimension is zero.

Column space is all of $\mathbb{R}^3$ and it is spanned by the column vectors of A.

3. (20 points) Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$.

(10 points bonus) Is it possible to diagonalize it? If so, complete the following: $T(x, y) = (?, ?)$ where x, y are the coordinates in the basis consisting of the eigenvectors.

Eigenvalues are calculated using the equation $det(A - \lambda I) = 0$.

$$det\left(\begin{bmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{bmatrix}\right) = 0 \implies (1-\lambda)(3-\lambda) - 8 = 0 \implies \lambda^2 - 4\lambda - 5 = 0.$$

Factoring, $(\lambda - 5)(\lambda + 1) = 0$. So eigenvalues are -1 and 5.

To find the eigenspace for -1, we find the null space of $A + I$: Start with the augmented matrix of $A + I = \mathbf{0}$.

$$\begin{bmatrix} 2 & 2 & 0 \\ 4 & 4 & 0 \end{bmatrix} \to \begin{bmatrix} 2 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Letting y to be free variable the eigenspace is $\left\{ y \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$.

To find the eigenspace for 5, we find the null space of $A-5I$: Start with the augmented matrix of $A-5I=\mathbf{0}$.

$$\begin{bmatrix} -4 & 2 & 0 \\ 4 & -2 & 0 \end{bmatrix} \to \begin{bmatrix} -4 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Letting x to be free variable the eigenspace is $\left\{ x \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$.

Both eigenspaces are of dimension 1 and they add up to 2 and thus this matrix is diagonalizable.

$T(x,y)=(-x,5y)$ in the basis consisting of the eigenvectors because $T(x\mathbf{v}_1+y\mathbf{v}_2)=xT(\mathbf{v}_1)+yT(\mathbf{v}_2)=-x\mathbf{v}_1+5y\mathbf{v}_2$ where $\mathbf{v}_1,\mathbf{v}_2$ are the eigenvalues for -1 and 5 respectively.

NOTE: We get independent eigenvectors for free because eigenvalues are distinct.

ALSO NOTE: The matrix $P=\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$ is not the matrix of T under the basis consisting of the eigenvectors. The matrix is actually $D=P^{-1}AP$ the diagonal matrix $\begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix}$.

$$\begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ 5y \end{bmatrix}.$$

Also, multiplying a vector by P gives the co-ordinates of that vector in the standard basis. It doesn't give the image of that vector under T.

4. (20 points) Solve $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$ by finding the inverse of the matrix on the left. The solution is the coordinate vector of $\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$ in what basis?

   (Bonus 10 points) Solve the same using Cramer's rule. Your answers must be the same.

   Solution: After row reduction you get

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & -1 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1/2 & 1/2 & 1/2 \end{bmatrix}.$$

$$\to \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1/2 & 1/2 & -1/2 \\ 0 & 0 & 1 & -1/2 & 1/2 & 1/2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & 1/2 & -1/2 & 1/2 \\ 0 & 1 & 0 & 1/2 & 1/2 & -1/2 \\ 0 & 0 & 1 & -1/2 & 1/2 & 1/2 \end{bmatrix}$$

$$So \quad A^{-1} = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & 1/2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix} \; and \; \begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1}\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 1/2 \\ 5/2 \end{bmatrix}$$

   This is the coordinate vector in the basis consisting of the column vectors.

   Using Cramer's rule

   It is easy to calculate the determinant of given matrix (using any method)by expanding along first column: $1(1-0)+1(1-0)=2$. So the solution is

$$x=\frac{1}{2}\begin{vmatrix} 2 & 1 & 0 \\ 3 & 1 & 1 \\ 4 & 0 & 1 \end{vmatrix}=\frac{3}{2}\;;\; y=\frac{1}{2}\begin{vmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \\ 1 & 4 & 1 \end{vmatrix}=\frac{1}{2}\;;\; z=\frac{1}{2}\begin{vmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 1 & 0 & 4 \end{vmatrix}=\frac{5}{2}$$

5. (20 points) Show that the set W of *differentiable* real valued functions is a subspace of the vector space V of all real valued functions on the real number line. [You only need to verify the requirements for a subspace].

(b) Show that if two functions $f, g \in W$ are linearly dependent then $(f'g - g'f)(t) = 0$ for all $t \in \mathbb{R}$. Thus if $(f'g - g'f)(t) \neq 0$ even for a single $t \in \mathbb{R}$ then f, g are linearly independent.

(c) Use (b) or any method to find a basis for the subspace spanned by $\{1, \cos^2 t, \sin^2 t\}$.

Solution

The sum of two differentiable functions as well as scalar multiples of differentiable functions are also differentiable. The zero function is differentiable. So W is a subspace.

If $f = kg$ then $f'g - fg' = kg'g - kgg' = 0$ for all real values.

Since $\cos^2 t + \sin^2 t = 1$, they are dependent. So we can leave out one of them. Looking at $f = 1$ and $g = \cos^2 t$, we have $f'g - fg' = 2 \cos t \sin t$ which is nonzero for any value between 0 and $\pi/2$. So they are independent and form a basis for the subspace spanned by $\{1, \cos^2 t, \sin^2 t\}$ because any function of the form $a + b\cos^2 t + c\sin^2 t$ can be expressed as a combination of $f = 1$ and $g = \cos^2 t$, and they are linearly independent.