1. (20 points) Say whether the following are true or false. If true prove it. If false give a counter-example.

   (a) If an $n \times n$ matrix A is invertible and $AB = I_n$ for another $n \times n$ matrix B then $BA = I_n$ as well.
   (b) If a system of linear equations represented by $x_1\mathbf{v_1} + x_2\mathbf{v_2} + ... + x_n\mathbf{v_n} = \mathbf{a}$ has a solution for every $\mathbf{a}$, then the vectors $\mathbf{v}_i,\ i = 1, 2, ..., n$ are linearly independent.

   Solution

   1a. True. Multiplying both sides of $AB = I_n$ *on the left* by A^{-1} we get $B = A^{-1}$ and thus by definition $BA = I_n$ as well.

   1b. False. This only means the corresponding matrix gives rise to a linear map that is onto. For example you can make a 2×3 matrix with one free variable but two pivot rows and it will give an onto linear map. The free variable means homogenous equation has nontrivial solutions which means the linear map is not one-one which means column vectors $\mathbf{v}_i$ are dependent.

2. (20 points) The first matrix A is row equivalent to the second matrix. Find the basis for the columns space Col A and the basis for the null space Nul A. What are the rank and nullity?

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

   Solution

   Column Space

   The first two columns $\mathbf{b}_1, \mathbf{b}_2$ of the reduced echelon matrix are independent. The last two are dependent on the first two. In fact, $\mathbf{b}_3 = -\mathbf{b}_1 + 2\mathbf{b}_2, \mathbf{b}_4 = -2\mathbf{b}_1 + 3\mathbf{b}_2$.

   Check that the columns of A also satisfy the same relationships! So the first two columns of A give a basis for Col A and the rank is 2.

   Null Space

   There are two free variables and the solution of homogenous equation is $x = z + 2w, y = -2z - 3w$.

   So we get the following two basis vectors for the null space and nullity is 2:

$$\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}.$$

3. (20 points) Check whether the map that sends (x_1, x_2, x_3) to $(x_1, 2x_2, x_2+x_3)$ is a linear transformation, in two ways. Not enough to check $T(\mathbf{0}) = \mathbf{0}$!

   (a) Check using definition of linear transformation.

   (b) By producing a matrix of the linear transformation using $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$.. Here $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and so on.

   Solution for (a)

   By the definition of the transformation we have $T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ 2x_2 \\ x_2 + x_3 \end{bmatrix}$

Let m, n be scalars and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ be ANY two vectors.

Using the above definition we have $T(\mathbf{u}) = \begin{bmatrix} u_1 \\ 2u_2 \\ u_2 + u_3 \end{bmatrix}$ and $T(\mathbf{v}) = \begin{bmatrix} v_1 \\ 2v_2 \\ v_2 + v_3 \end{bmatrix}$

Enough to show that $T(m\mathbf{u} + n\mathbf{v}) = mT(\mathbf{u}) + nT(\mathbf{v})$.

We have $T(m\mathbf{u} + n\mathbf{v}) = T \begin{bmatrix} mu_1 + nv_1 \\ mu_2 + nv_2 \\ mu_3 + nv_3 \end{bmatrix} = \begin{bmatrix} mu_1 + nv_1 \\ 2mu_2 + 2nv_2 \\ (mu_2 + nv_2) + (mu_3 + nv_3) \end{bmatrix}$

On the other hand $mT(\mathbf{u}) + nT(\mathbf{v}) = m \begin{bmatrix} u_1 \\ 2u_2 \\ u_2 + u_3 \end{bmatrix} + n \begin{bmatrix} v_1 \\ 2v_2 \\ v_2 + v_3 \end{bmatrix} = \begin{bmatrix} mu_1 + nv_1 \\ 2mu_2 + 2nv_2 \\ (mu_2 + mu_3) + (nv_2 + nv_3) \end{bmatrix}$

Comparing the two we see that they are equal and so $T(m\mathbf{u} + n\mathbf{v}) = mT(\mathbf{u}) + nT(\mathbf{v})$.

Solution for (b)

The matrix A of the linear transformation is given below:

$$T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

The image vector $T(\mathbf{e}_1)$ is given by $T \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2(0) \\ 0+0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

The image vector $T(\mathbf{e}_2)$ is given by $T \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1+0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$.

The image vector $T(\mathbf{e}_3)$ is given by $T \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2(0) \\ 0+1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Putting these together as columns in one matrix we get the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$.

4. (20 points) Suppose a nation A sends 0.1 of its population to nation B each year and remaining 0.9 stay home. Nation B sends 0.2 of its population to A and 0.8 stay home. What are the populations that will stay stable (instead of one country steadily increasing relative to the other)? (say A has 100 units, then what is population of B).

   Solution: Let A and B be their populations.

   Get two equations if the populations stay stable:

$$\begin{aligned} 0.9A + 0.2B &= A \\ 0.1A + 0.8B &= B \end{aligned}$$

   Simplifying, we get

$$\begin{aligned} -0.1A + 0.2B &= 0 \\ 0.1A - 0.2B &= 0 \end{aligned}$$

This is really just one equation, and we get $0.1A = 0.2B \implies A = 2B$ in order for the populations to stay stable.

So A will have 100 units, B will have 50 units.

Basically, we need that 0.1 of A equals 0.2 of B so that what A loses it gains back from B and vice versa.

5. (20 points) Find the inverse of the following matrix, if it exists. If A^{-1} exists, check that AA^{-1} is the identity matrix.

$$\begin{bmatrix} 0 & 6 & 0 \\ 0 & 0 & 1 \\ 4 & 0 & 0 \end{bmatrix}$$

Solution

Solution using row reduction. Need to get identity matrix on left.

Remember that the row operations have to be performed throughout.

$$\begin{bmatrix} 0 & 6 & 0 & | & 1 & 0 & 0 \\ 0 & 0 & 1 & | & 0 & 1 & 0 \\ 4 & 0 & 0 & | & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_3 \leftrightarrow R_1} \begin{bmatrix} 4 & 0 & 0 & | & 0 & 0 & 1 \\ 0 & 0 & 1 & | & 0 & 1 & 0 \\ 0 & 6 & 0 & | & 1 & 0 & 0 \end{bmatrix} \xrightarrow{R_3 \leftrightarrow R_2} \begin{bmatrix} 4 & 0 & 0 & | & 0 & 0 & 1 \\ 0 & 6 & 0 & | & 1 & 0 & 0 \\ 0 & 0 & 1 & | & 0 & 1 & 0 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{4}R_1 \to R_1} \begin{bmatrix} 1 & 0 & 0 & | & 0 & 0 & \frac{1}{4} \\ 0 & 6 & 0 & | & 1 & 0 & 0 \\ 0 & 0 & 1 & | & 0 & 1 & 0 \end{bmatrix} \xrightarrow{\frac{1}{6}R_2 \to R_2} \begin{bmatrix} 1 & 0 & 0 & | & 0 & 0 & \frac{1}{4} \\ 0 & 1 & 0 & | & \frac{1}{6} & 0 & 0 \\ 0 & 0 & 1 & | & 0 & 1 & 0 \end{bmatrix}$$

The matrix on the right, namely $\begin{bmatrix} 0 & 0 & \frac{1}{4} \\ \frac{1}{6} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ is the inverse matrix.

Check that it is correct by multiplying with the original matrix. You must get the identity matrix.

6. (Challenge, 20 points) Show that the set of vectors (x, y, z) whose co-ordinates satisfy $x + 4y - 5z = 0$ is a subspace of $\mathbb{R}^3$. What is its dimension? Prove your answer by producing a basis for the subspace.

An easy way to show this is to see that this is the null space of the map given by $[1\ 4\ -5]\mathbf{v}$. i.e, the matrix of the transformation is $[1\ 4\ -5]$.

It will have two free variables, so dimension is 2. $x = -4y + 5z$ means the null space is spanned by the vectors $\begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}$.

Alternatively, you can use the subspace definition: It needs to be closed under addition and scalar multiplication, and must have zero vector in it.

Clearly the zero vector is a solution.

If you take any two vectors satisfying this equation their sum is also a solution. Any scalar multiple is also a solution:

$$x_1 + 4y_1 - 5z_1 = 0 \text{ and } x_2 + 4y_2 - 5z_2 = 0 \implies (x_1 + x_2) + 4(y_1 + y_2) - 5(z_1 + z_2) = 0.$$

$$x + 4y - 5z = 0 \implies cx + 4cy - 5cz = 0 \quad \textit{for any } c.$$

You may recognize this as the equation of a plane, so its dimension is 2. Need two linearly independent vectors on the plane, meaning two vectors that satisfy the equation and are linearly independent. Easy to see that the two vectors above form such a basis set.