1. (20 points) Say whether the following are true or false. If true prove it. If false give a counter-example.

   (a) $\mathbb{R}^3$ is a subspace of $\mathbb{R}^4$.

   (b) A is a 3×3 matrix. If A is diagonalizable, then any basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ of $\mathbb{R}^3$ consists of eigenvectors, i.e, $T(\mathbf{v}_i) = \lambda_i \mathbf{v}_i$ for some $\lambda_i, i = 1, 2, 3$

   Solution

   1a. False. $\mathbb{R}^3$ is not even a subset of $\mathbb{R}^4$ because you have different number of coordinates.

   1b. False. This is only true for the basis consisting of the eigenvectors.

2. (30 points) Let A be an $n \times n$ diagonalizable matrix, with P having the eigenvectors $\mathbf{v}_j, j = 1, 2, ..., n$ in its columns and D the diagonal matrix with the eigenvalues, so that $A = PDP^{-1}$. Let λ_j be the eigenvalues corresponding to $\mathbf{v}_j$.

   a) Show that $det(A) = det(D)$. Thus the determinant of A is the product of its eigenvalues (counting multiplicities). (First show that $det(P^{-1}) = 1/det(P)$ using $PP^{-1} = I_n$.)

   b) (Really should have said $A^k = PD^kP^{-1}$). Show that $det(A^k) = det(D^k)$ for $k = 2, 3, ...,$ where $A^k = A \times A \times .... \times A$ (the product of the matrix with itself k times).

   (c) $A\mathbf{x}_i = \mathbf{x}_{i+1}$ is the equation for a Markov chain, where $\mathbf{x}_i$ is the population vector after i years, then show that the population after $i + k$ years is $\mathbf{x}_{i+k} = A^k\mathbf{x}_i = \begin{bmatrix} \lambda_1^k x_{i,1} \\ \lambda_2^k x_{i,2} \\ . \\ . \\ \lambda_n^k x_{i,n} \end{bmatrix}$ where $\{x_{i,j}\}$ is the coordinate set of $\mathbf{x}_i$ in the eigenvector basis. That is, $\mathbf{x}_i = \sum_{j=1}^n x_{i,j}\mathbf{v}_j$ and $\mathbf{x}_{i+k} = \sum_{j=1}^n \lambda_j^k x_{i,j}\mathbf{v}_j$

   Solution

   You can also do (a) and (b) using only the multiplicativity of determinants.

   For example, in (a) below you can say $det(P)det(D)det(P^{-1}) = det(D)det(P)det(P^{-1}) = det(D)det(PP^{-1}) = det(D)det(I) = det(D)$.

   In (b) below you can say $det(A) = det(D) \implies det(A^k) = (det(A))^k = (det(D))^k = det(D^k)$.

   a) $PP^{-1} = I_n \implies det(PP^{-1}) = det(P)det(P^{-1}) = det(I_n) = 1 \implies det(P^{-1}) = 1/det(P)$. $det(PDP^{-1}) = det(P)det(D)det(P^{-1}) = det(P)det(D)/det(P) = det(D)$.

   b) $A^2 = (PDP^{-1})(PDP^{-1}) = PD(P^{-1}P)DP^{-1} = PDI_nDP^{-1} = PD^2P^{-1}$. Proceeding like this, we get $A^k = PD^kP^{-1}$ and $det(A^k) = det(D^k)$ similar to how we proved (a).

   c) $A^k\mathbf{x}_i = D^k[\mathbf{x}_i]_B = \sum_{j=1}^n \lambda_j^k x_{i,j}\mathbf{v}_j$ where B is the basis consisting of the eigenvectors. You can also do this by applying A^k to both sides of $\mathbf{x}_i = \sum_{j=1}^n x_{i,j}\mathbf{v}_j$ and then using $A^k\mathbf{v}_j = \lambda_j^k\mathbf{v}_j$. You can get the last equation inductively, i.e, $A^2\mathbf{v}_j = A(A\mathbf{v}_j) = A(\lambda_j\mathbf{v}_j) = \lambda_j^2\mathbf{v}_j \implies A^3\mathbf{v}_j = A(\lambda_j^2\mathbf{v}_j) = \lambda^3\mathbf{v}_j \implies .... \implies A^k\mathbf{v}_j = \lambda_j^k\mathbf{v}_j$.

3. (30 points) Solve by using Cramer's rule.

   First show that the determinant of the matrix on the left is 3 :

$$\begin{aligned} x + y - z &= 1 \\ x + 2y - z &= 2 \\ 2x - y + z &= 3 \end{aligned}$$

Solution: Writing in matrix form you get

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & -1 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Using Cramer's rule

It is easy to calculate the determinant of given matrix by expanding along first column: $1(2-1) - 1(1-1) + 2(-1+2) = 3$. So the solution is

$$x = \frac{1}{3}\begin{vmatrix} 1 & 1 & -1 \\ 2 & 2 & -1 \\ 3 & -1 & 1 \end{vmatrix} = \frac{4}{3}\ ;\ y = \frac{1}{3}\begin{vmatrix} 1 & 1 & -1 \\ 1 & 2 & -1 \\ 2 & 3 & 1 \end{vmatrix} = \frac{3}{3} = 1\ ;\ z = \frac{1}{3}\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 2 & -1 & 3 \end{vmatrix} = \frac{4}{3}$$

Check that this satisfies given equation.

4. (20 points) (a) Show that the matrix of the problem (3) above is invertible *without calculating its echelon matrix.*

   (b) Show that the matrix of the problem (3) above is one to one *without calculating its echelon matrix.*

   (c) What is its null space?

   (d) Show that the subspace spanned by its column vectors (i.e, the column space) is all of $\mathbb{R}^3$.

Solution

It is invertible because determinant is nonzero.

If it is invertible, then homogenous equation has only zero vector as solution because $A\mathbf{v} = \mathbf{0} \implies \mathbf{v} = A^{-1}\mathbf{0} = \mathbf{0}$. So it is one-one. You can also say that problem 3 shows that every equation has unique solution because we can use same method as in problem 3 to find the solution.

Null space is $\{\mathbf{0}\}$ because it is one-one. Only zero vector is in the solution space (null space).

Because it is one-one, its column vectors are independent. So Column space is 3 dimensional and hence $\mathbb{R}^3$. This is because any set of 3 independent vectors in a 3 dimensional space is also a basis. So its column vectors form a basis for $\mathbb{R}^3$ and hence their span is all of $\mathbb{R}^3$.