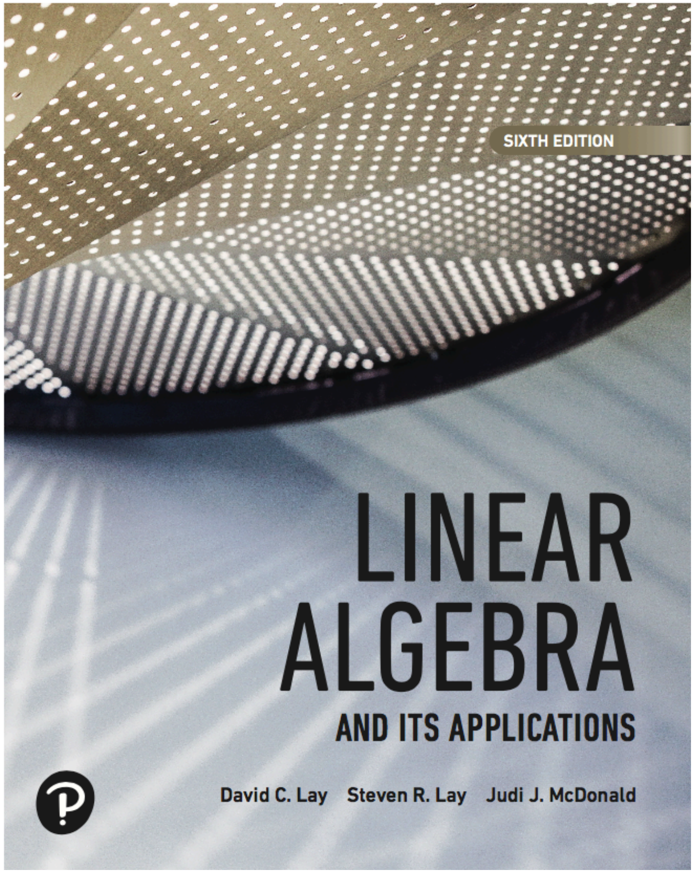
1

Linear Equations in Linear Algebra

1.2

Row Reduction and Echelon Forms



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ECHELON FORM

- A rectangular matrix is in **echelon form** (or **row echelon form**) if it has the following three properties:
  1. All nonzero rows are above any rows of all zeros.
  2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
  3. All entries in a column below a leading entry are zeros.

WHAT DOES AN ECHELON FORM DO?

- Echelon form basically helps us to solve the system using back substitution:
    1. If number of rows = number of columns (i.e, number of equations = number of variables), then you end up with a triangular matrix and we can start solving with the last variable.
    2. If there are more columns than rows (more variables than equations) then you get free variables and families of solutions.

ECHELON FORM

- If a matrix in echelon form satisfies the following additional conditions, then it is in **reduced echelon form** (or **reduced row echelon form**):

   4. The leading entry in each nonzero row is 1.
   5. Each leading 1 is the only nonzero entry in its column.

- An **echelon matrix** (respectively, **reduced echelon matrix**) is one that is in echelon form (respectively, reduced echelon form.)

ECHELON FORM

- Any nonzero matrix may be **row reduced** (i.e., transformed by elementary row operations) into more than one matrix in echelon form, using different sequences of row operations. However, the reduced echelon form one obtains from a matrix is unique.

  Theorem 1: Uniqueness of the Reduced Echelon Form

  Each matrix is row equivalent to one and only one reduced echelon matrix.

WHY IS REDUCED ECHELON FORM UNIQUE?

- We will see the proof later, after we have studied vector spaces.
- But the basic idea is that the final solution is unique, and since the reduced echelon form gives the final solution, it must be unique.
- Geometrically, for example, two planes in 3 dimensions can intersect in a plane or a line or nowhere. In each case there is only one way to write the solution.

PIVOT POSITION

- If a matrix A is row equivalent to an echelon matrix U, we call U **an echelon form** (or row echelon form) **of** A; if U is in reduced echelon form, we call U **the reduced echelon form of** A.

- A **pivot position** in a matrix A is a location in A that corresponds to a leading 1 in the reduced echelon form of A. A **pivot column** is a column of A that contains a pivot position.

IMPORTANCE OF PIVOT POSITION

- The **pivot position** in a matrix A in the reduced echelon form determines what kind of solutions we have.
- For example, if number of rows = number of columns and the reduced echelon form has 1's in all the diagonal positions, then the system will have a unique solution.
- The total number of columns (other than the last one) minus the number of pivot columns (ones with pivot entries) gives number of free variables.

PIVOT POSITION

- **Example 1:** Row reduce the matrix A below to echelon form, and locate the pivot columns of A.

$$A = \begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

- **Solution:** The top of the leftmost nonzero column is the first pivot position. A nonzero entry, or *pivot*, must be placed in this position.

PIVOT POSITION

- Now, interchange rows 1 and 4.

Pivot

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

Pivot column

- Create zeros below the pivot, 1, by adding multiples of the first row to the rows below, and obtain the next matrix.

PIVOT POSITION

- Choose 2 in the second row as the next pivot.

Pivot

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

Next pivot column

- Add $-5/2$ times row 2 to row 3, and add $3/2$ times row 2 to row 4.

PIVOT POSITION

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

- There is no way a leading entry can be created in column 3. But, if we interchange rows 3 and 4, we can produce a leading entry in column 4.

PIVOT POSITION

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 \xleftarrow{\text{Pivot}} & 0 \\ \underset{\uparrow}{0} & \underset{\uparrow}{0} & 0 & \underset{\uparrow}{0} & 0 \end{bmatrix} \text{Pivot columns}$$

- The matrix is in echelon form and thus reveals that columns 1, 2, and 4 of A are pivot columns.

PIVOT POSITION

Pivot positions

$$A = \begin{bmatrix} \boxed{0} & -3 & -6 & 4 & 9 \\ -1 & \boxed{-2} & -1 & 3 & 1 \\ -2 & -3 & 0 & \boxed{3} & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

Pivot columns

- The pivots in the example are 1, 2 and -5.

ROW REDUCTION ALGORITHM

- **Example 2:** Apply elementary row operations to transform the following matrix first into echelon form and then into reduced echelon form.

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

- **Solution:**
- **STEP 1:** Begin with the leftmost nonzero column. This is a pivot column. The pivot position is at the top.

ROW REDUCTION ALGORITHM

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

Pivot column

- **STEP 2:** Select a nonzero entry in the pivot column as a pivot. If necessary, interchange rows to move this entry into the pivot position.

ROW REDUCTION ALGORITHM

- Interchange rows 1 and 3. (Rows 1 and 2 could have also been interchanged instead.)

Pivot

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

- **STEP 3:** Use row replacement operations to create zeros in all positions below the pivot.

ROW REDUCTION ALGORITHM

- We could have divided the top row by the pivot, 3, but with two 3s in column 1, it is just as easy to add -1 times row 1 to row 2.

Pivot

$$\begin{bmatrix} \overset{\text{Pivot}}{3} & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

- **STEP 4:** Cover the row containing the pivot position, and cover all rows, if any, above it. Apply steps 1–3 to the submatrix that remains. Repeat the process until there are no more nonzero rows to modify.

ROW REDUCTION ALGORITHM

- With row 1 covered, step 1 shows that column 2 is the next pivot column; for step 2, select as a pivot the “top” entry in that column.

Pivot

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

New pivot column

- For step 3, we could insert an optional step of dividing the “top” row of the submatrix by the pivot, 2. Instead, we add $-3/2$ times the “top” row to the row below.

ROW REDUCTION ALGORITHM

- This produces the following matrix.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

- When we cover the row containing the second pivot position for step 4, we are left with a new submatrix that has only one row.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

ROW REDUCTION ALGORITHM

- Steps 1–3 require no work for this submatrix, and we have reached an echelon form of the full matrix. We perform one more step to obtain the reduced echelon form.
- **STEP 5:** Beginning with the rightmost pivot and working upward and to the left, create zeros above each pivot. If a pivot is not 1, make it 1 by a scaling operation.
- The rightmost pivot is in row 3. Create zeros above it, adding suitable multiples of row 3 to rows 2 and 1.

ROW REDUCTION ALGORITHM

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 2 & -4 & 4 & 0 & -14 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \begin{array}{l} \leftarrow \text{Row } 1 + (-6) \times \text{row } 3 \\ \leftarrow \text{Row } 2 + (-2) \times \text{row } 3 \\ \end{array}$$

- The next pivot is in row 2. Scale this row, dividing by the pivot.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \begin{array}{l} \\ \leftarrow \text{Row scaled by } \dfrac{1}{2} \\ \end{array}$$

ROW REDUCTION ALGORITHM

- Create a zero in column 2 by adding 9 times row 2 to row 1.

$$\begin{bmatrix} 3 & 0 & -6 & 9 & 0 & -72 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \longleftarrow \text{Row } 1 + (9) \times \text{row } 2$$

- Finally, scale row 1, dividing by the pivot, 3.

ROW REDUCTION ALGORITHM

$$\begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \begin{matrix} \longleftarrow \text{Row scaled by } \frac{1}{3} \\ \\ \\ \end{matrix}$$

- This is the reduced echelon form of the original matrix.
- *Notice that reduced echelon is not ALL 1's and 0's. If there are no free variables, then it would be all 1's and 0's (other than last column). Basically, ONLY THE PIVOT COLUMNS are all 1's and 0's.*

- The combination of steps 1–4 is called the **forward phase** of the row reduction algorithm. Step 5, which produces the unique reduced echelon form, is called the **backward phase**.

SOLUTIONS OF LINEAR SYSTEMS

- The row reduction algorithm leads to an explicit description of the solution set of a linear system when the algorithm is applied to the augmented matrix of the system.
- Suppose that the augmented matrix of a linear system has been changed into the equivalent *reduced* echelon form.

$$\begin{bmatrix} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

SOLUTIONS OF LINEAR SYSTEMS

- There are 3 variables because the augmented matrix has four columns. The associated system of equations is

$$\begin{aligned} x_1 - 5x_3 &= 1 \\ x_2 + x_3 &= 4 \\ 0 &= 0 \end{aligned} \qquad \text{----(1)}$$

- The variables x_1 and x_2 corresponding to pivot columns in the matrix are called **basic variables**. The other variable, x_3, is called a **free variable**.

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SOLUTIONS OF LINEAR SYSTEMS

- Whenever a system is consistent, as in (1), the solution set can be described explicitly by solving the *reduced* system of equations for the basic variables in terms of the free variables.

- This operation is possible because the reduced echelon form places each basic variable in one and only one equation.

- In (1), solve the first and second equations for x_1 and x_2. (Ignore the third equation; it offers no restriction on the variables.)

SOLUTIONS OF LINEAR SYSTEMS

$$\begin{aligned} x_1 &= 1 + 5x_3 \\ x_2 &= 4 - x_3 \qquad \text{----(2)} \\ x_3 &\ \text{is free} \end{aligned}$$

- The statement "x_3 is free" means that you are free to choose any value for x_3. Once that is done, the formulas in (2) determine the values for x_1 and x_2.
- For instance, when $x_3 = 0$, the solution is (1,4,0); and the solution is (6,3,1) when $x_3 = 1$.

- *Each different choice of* x_3 *determines a (different) solution of the system, and every solution of the system is determined by a choice of* x_3.

REASONABLE ANSWERS

- We can now verify that our solution is “reasonable” for a given matrix. Write down the system of equations associate with the matrix:

$$\begin{bmatrix} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Longrightarrow \begin{aligned} x_1 - 5x_3 &= 1 \\ x_2 + x_3 &= 4 \\ 0 &= 0 \end{aligned}$$

REASONABLE ANSWERS

- Then substitute in the solution you found for each variable and verify that the equations on the left add to the correct amount:

$$\begin{aligned} x_1 - 5x_3 &= 1 \\ x_2 + x_3 &= 4 \\ 0 &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} (1 + 5x_3) - 5x_3 &= 1 \\ (4 - x_3) + x_3 &= 4 \\ 0 &= 0 \end{aligned}$$

PARAMETRIC DESCRIPTIONS OF SOLUTION SETS

- The description in (2) is a *parametric description* of solutions sets in which the free variables act as parameters.

- *Solving a system* amounts to finding a parametric description of the solution set or determining that the solution set is empty.

- Whenever a system is consistent and has free variables, the solution set has many parametric descriptions.

PARAMETRIC DESCRIPTIONS OF SOLUTION SETS

- For instance, in system (1), add 5 times equation 2 to equation 1 and obtain the following equivalent system.

$$x_1 + 5x_2 = 21$$

$$x_2 + x_3 = 4$$

- We could treat x_2 as a parameter and solve for x_1 and x_3 in terms of x_2, and we would have an accurate description of the solution set.
- When a system is inconsistent, the solution set is empty, even when the system has free variables. In this case, the solution set has no parametric representation.

EXISTENCE AND UNIQUENESS THEOREM

Theorem 2: Existence and Uniqueness Theorem

A linear system is consistent if and only if the rightmost column of the augmented matrix is *not* a pivot column—i.e., if and only if an echelon form of the augmented matrix has *no* row of the form

$$[0 \;\ldots\; 0 \;\; b] \text{ with } b \text{ nonzero.}$$

- If a linear system is consistent, then the solution set contains either (i) a unique solution, when there are no free variables, or (ii) infinitely many solutions, when there is at least on free variable.

ROW REDUCTION TO SOLVE A LINEAR SYSTEM

Using Row Reduction to Solve a Linear System

1. Write the augmented matrix of the system.
2. Use the row reduction algorithm to obtain an equivalent augmented matrix in echelon form. Decide whether the system is consistent. If there is no solution, stop; otherwise, go to the next step.
3. Continue row reduction to obtain the reduced echelon form.
4. Write the system of equations corresponding to the matrix obtained in step 3.

ROW REDUCTION TO SOLVE A LINEAR SYSTEM

5. Rewrite each nonzero equation from step 4 so that its one basic variable is expressed in terms of any free variables appearing in the equation.