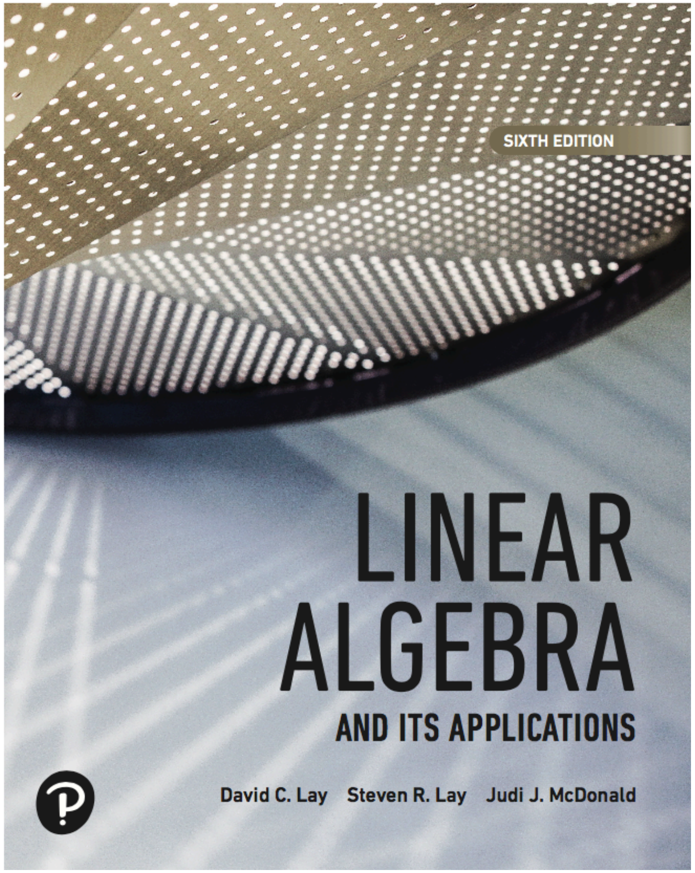
1 Linear Equations in Linear Algebra

1.6

APPLICATIONS OF LINEAR SYSTEMS



A HOMOGENEOUS SYSTEM IN MIGRATION

- Suppose a nation A sends 0.2 of its population to nation B each year and remaining 0.8 stay home. Nation B sends 0.3 of its population to A and 0.7 stay home. What are the populations that will stay stable? (instead of one country steadily increasing relative to the other?)

A HOMOGENEOUS SYSTEM IN MIGRATION

We get two equations:
$$\begin{aligned} 0.2A + 0.7B &= B \\ 0.8A + 0.3B &= A \end{aligned}$$

These can be converted to homogenous equations :

- $$\begin{aligned} 0.2A - 0.3B &= 0 \\ -0.2A + 0.3B &= 0 \end{aligned}$$
- You can see that these equations have solutions.
- Basically we need B = (0.2/0.3)A = 0.66A approx

in order for the populations to be stable.

A HOMOGENEOUS SYSTEM IN ECONOMICS

- Suppose a nation's economy is divided into many sectors.
- Suppose that for each sector we know its total output for one year and we know exactly how this output is divided or "exchanged" among the other sectors of the economy.
- Let the total dollar value of a sector's output be called the **price** of that output.

A HOMOGENEOUS SYSTEM IN ECONOMICS

- Leontief proved the following result:

There exist *equilibrium prices* that can be assigned to the total outputs of the various sectors in such a way that the income of each sector exactly balances its expenses.

A HOMOGENEOUS SYSTEM IN ECONOMICS

Suppose an economy consists of the Coal, Electric (power), and Steel sectors, and the output of each sector is distributed among the various sectors as in Table 1

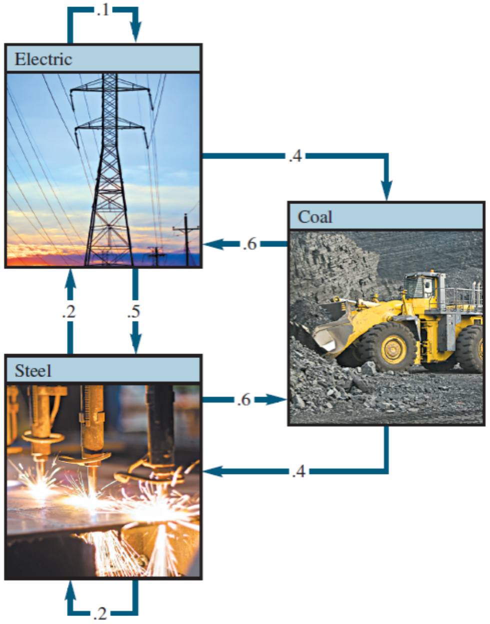


TABLE 1 A Simple Economy

Distribution of Output from			
Coal	**Electric**	**Steel**	**Purchased by**
.0	.4	.6	Coal
.6	.1	.2	Electric
.4	.5	.2	Steel

A HOMOGENEOUS SYSTEM IN ECONOMICS

- Denote the prices of the total annual outputs of the Coal, Electric, and Steel sectors by p_C, p_E, and $p_{S.}$
- A sector looks down a column to see where its output goes, and it looks across a row to see what it needs as inputs.

A HOMOGENEOUS SYSTEM IN ECONOMICS

- For instance, the first row of Table 1 says that Coal receives (and pays for) 40% of the Electric output and 60% of the Steel output.
- Coal must spend .4 p_E dollars for its share of Electric's output and .6 p_S or its share of Steel's output.

$$p_C = .4\,p_E + .6\,p_S$$

A HOMOGENEOUS SYSTEM IN ECONOMICS

- The Electric sector spends .6 p_C for coal, .1 p_E for electricity, and .2 p_S for steel.

$$p_E = .6\,p_C + .1\,p_E + .2\,p_S$$

- The third row of the exchange table leads to:

$$p_S = .4\,p_C + .5\,p_E + .2\,p_S$$

A HOMOGENEOUS SYSTEM IN ECONOMICS

- To solve the system of equations move all the unknowns to the left sides of the equations and combine like terms:

$$\begin{aligned} p_{\mathrm{C}} \quad - \quad .4\,p_{\mathrm{E}} \quad - \quad .6\,p_{\mathrm{S}} &= 0 \\ -.6\,p_{\mathrm{C}} \quad + \quad .9\,p_{\mathrm{E}} \quad - \quad .2\,p_{\mathrm{S}} &= 0 \\ -.4\,p_{\mathrm{C}} \quad - \quad .5\,p_{\mathrm{E}} \quad + \quad .8\,p_{\mathrm{S}} &= 0 \end{aligned}$$

A HOMOGENEOUS SYSTEM IN ECONOMICS

- Putting in a matrix and row reducing we get

$$\begin{bmatrix} 1 & -.4 & -.6 & 0 \\ -.6 & .9 & -.2 & 0 \\ -.4 & -.5 & .8 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -.94 & 0 \\ 0 & 1 & -.85 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} p_{\mathbf{C}} \\ p_{\mathbf{E}} \\ p_{\mathbf{S}} \end{bmatrix} = \begin{bmatrix} .94 p_{\mathbf{S}} \\ .85 p_{\mathbf{S}} \\ p_{\mathbf{S}} \end{bmatrix} = p_{\mathbf{S}} \begin{bmatrix} .94 \\ .85 \\ 1 \end{bmatrix}$$

A HOMOGENEOUS SYSTEM IN ECONOMICS

- Any positive choice for p_S results in a choice of equilibrium prices.
- For instance, if we set the price $p_S = \$100$ million, then the incomes and expenditures of each sector will be equal if the output of Coal is priced at \$94 million, that of Electric at \$85 million, and that of Steel at \$100 million.

BALANCING CHEMICAL EQUATIONS

- Chemical equations describe the quantities of substances consumed and produced by chemical reactions. For instance, when propane gas burns:

$$(x_1)C_3H_8 + (x_2)O_2 \rightarrow (x_3)CO_2 + (x_4)H_2O$$

- To "balance" this equation, a chemist must find whole numbers $x_1, \ldots, x_4$ such that the total numbers of carbon (C), hydrogen (H), and oxygen (O) atoms on the left match the corresponding numbers of atoms on the right.

BALANCING CHEMICAL EQUATIONS

- Expressing each chemical as a vector:

$$C_3H_8:\begin{bmatrix}3\\8\\0\end{bmatrix},\quad O_2:\begin{bmatrix}0\\0\\2\end{bmatrix},\quad CO_2:\begin{bmatrix}1\\0\\2\end{bmatrix},\quad H_2O:\begin{bmatrix}0\\2\\1\end{bmatrix}\begin{matrix}\leftarrow & \text{Carbon}\\ \leftarrow & \text{Hydrogen}\\ \leftarrow & \text{Oxygen}\end{matrix}$$

- The chemical equation can be expressed as

$$x_1\begin{bmatrix}3\\8\\0\end{bmatrix}+x_2\begin{bmatrix}0\\0\\2\end{bmatrix}=x_3\begin{bmatrix}1\\0\\2\end{bmatrix}+x_4\begin{bmatrix}0\\2\\1\end{bmatrix}$$

BALANCING CHEMICAL EQUATIONS

- To solve, move all the terms to the left, and solve the augmented system to get

$$x_1 = \frac{1}{4}x_4,\ x_2 = \frac{5}{4}x_4,\ x_3 = \frac{3}{4}x_4, \text{with } x_4 \text{ free}$$

- Take $x_4 = 4$, in which case, $x_1 = 1$, $x_2 = 5$, and $x_3 = 3$. The balanced equation is

$$C_3H_8 + 5\ O_2 \rightarrow 3\ CO_2 + 4\ H_2O$$

NETWORK FLOW

- Systems of linear equations arise naturally when scientists, engineers, or economists study the flow of some quantity through a network.
- A *network* consists of a set of points called *nodes*, with lines called *branches* connecting some or all of the nodes. The direction of flow in each branch is indicated and the flow amount is shown.

NETWORK FLOW

- An example of a node

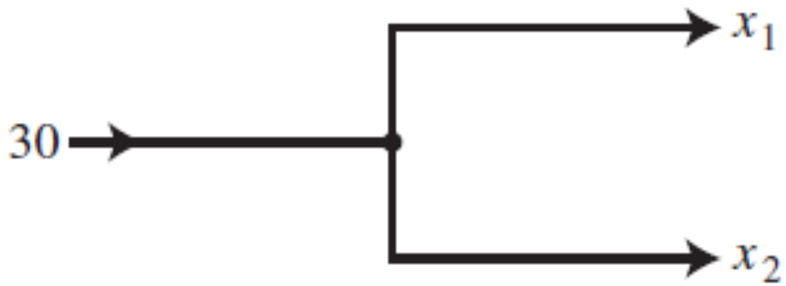


FIGURE 1

A junction or node.

NETWORK FLOW

- The network in Figure 2 shows the traffic flow over several one-way streets in downtown Baltimore. Determine the general flow pattern for the network.

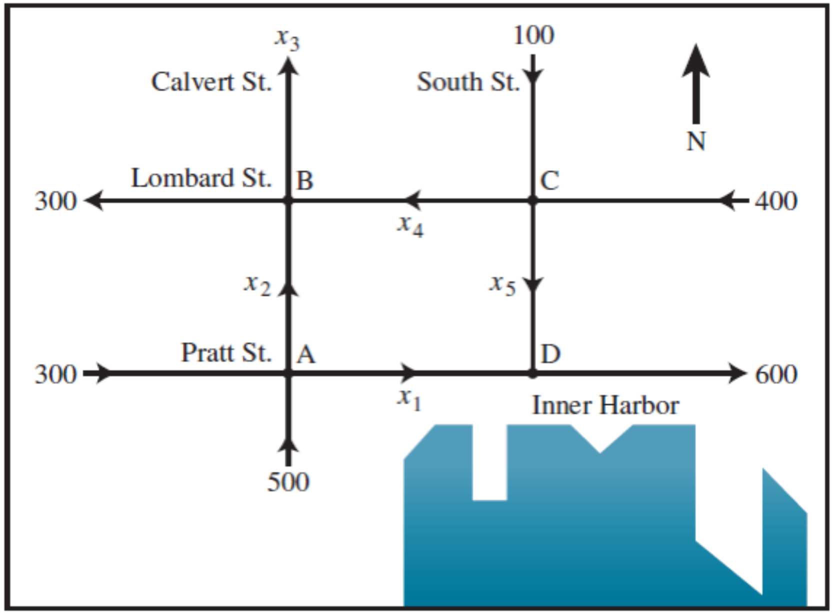


FIGURE 2 Baltimore streets.

NETWORK FLOW

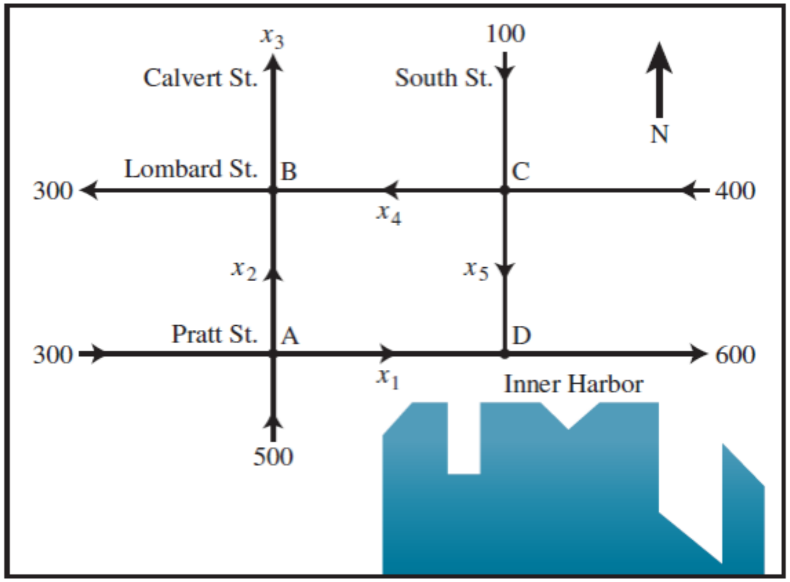


FIGURE 2 Baltimore streets.

Intersection	Flow in		Flow out
A	$300 + 500$	$=$	$x_1 + x_2$
B	$x_2 + x_4$	$=$	$300 + x_3$
C	$100 + 400$	$=$	$x_4 + x_5$
D	$x_1 + x_5$	$=$	600

NETWORK FLOW

$$
\begin{aligned}
x_1 + x_2 &= 800 \\
x_2 - x_3 + x_4 &= 300 \\
x_4 + x_5 &= 500 \\
x_1 + x_5 &= 600 \\
x_3 &= 400
\end{aligned}
$$

- The last equation comes from the equality

  Total flow out = Total flow in

  Namely,

  $x_3 + 600 + 300 = 400 + 100 + 500 + 300$

NETWORK FLOW

- Row reduction of the associated augmented matrix leads to

$$\begin{array}{cccccccc} x_1 & & & & + & x_5 & = & 600 \\ & x_2 & & & - & x_5 & = & 200 \\ & & x_3 & & & & = & 400 \\ & & & x_4 & + & x_5 & = & 500 \end{array} \qquad \begin{cases} x_1 = 600 - x_5 \\ x_2 = 200 + x_5 \\ x_3 = 400 \\ x_4 = 500 - x_5 \\ x_5 = x_5 \text{ is free} \end{cases}$$