Fall 2023 Notes, Differential Equations

Systems of Linear Differential Equations

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Outline

Matrices and Vector Equations

WILL RESTRICT TO 2 x 2 MATRICES.
CONCEPTS SIMILAR FOR BIGGER MATRICES.

The system of equations

$$ax + by = e$$
$$cx + dy = f$$

is written as

$$AX = C, \text{ where } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, C = \begin{pmatrix} e \\ f \end{pmatrix}, X = \begin{pmatrix} x \\ y \end{pmatrix}$$

A is a 2 x 2 matrix, C, X are 1 x 2 matrices. C, X are also vectors.

Systems of linear equations in matrix form

Recall the following from Chapter 3: A radioactive substance usually decays and becomes another radioactive substance or a stable (non-radioactive) substance.

Example: A decays into B at rate k_1, then B decays into the stable substance C at rate of k_2. We get the following system of first order ODE's: Amount of A is x, that of B is y and of C is z:

$$\frac{dx}{dt} = -k_1 x \qquad (1)$$

$$\frac{dy}{dt} = k_1 x - k_2 y \qquad (2)$$

$$\frac{dz}{dt} = k_2 y \qquad (3)$$

Example: Systems of linear equations in matrix form

The equations (1), (2) and (3) are simple enough we can solve them without using matrices as follows:
Solve (1) to get $x(t) = Ae^{-k_1 t}$ for some constant A.
Now plug in $x(t) = Ae^{-k_1 t}$ into (2) to get a linear equation $y' + Py = Q$ with $P = k_2, Q = k_1 Ae^{k_1 t}$ and solve it using the integrating factor method.
Finally plug in the solution for $y(t)$ thus obtained into (3) and integrate it.

But how to do this using matrices? Will see later.
First, we want to learn to write this in matrix form $AX = X'$:

$$\begin{pmatrix} -k_1 & 0 & 0 \\ k_1 & -k_2 & 0 \\ 0 & k_2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = X'(t).$$

Solving systems using matrices and eigenvalues

Suppose we are given a system of equation $x'(t) = \lambda x(t)$ and $y'(t) = \lambda y(t)$.
Then the solution is just $x(t) = ae^{\lambda t}, y(t) = be^{\lambda t}$ for some constants a, b.

In matrix form, the equation is $X'(t) = AX = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} X = \lambda X$.

The solution is $X = e^{\lambda t} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} ae^{\lambda t} \\ be^{\lambda t} \end{pmatrix}$.

Eigenvalues and eigenvectors help to solve matrix equations as if they were linear equations in matrices and vectors. i.e, *as if the matrix form of the differential equation is just* $X' = \lambda X$

Solving systems: eigenvalues

For a square, 2 x 2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ the determinant is $det(A) = ad - bc$. There is a general formula for bigger matrices but we will restrict to 2 x 2 matrices.

An **eigenvalue** λ is a real number such that $A\mathbf{v} = \lambda\mathbf{v}$.

A *non-zero* vector **v** for which eigenvalue exists is called **eigenvector.**

To find eigenvalues and eigenvectors we solve the *characteristic* equation $det(A - I\lambda) = 0$ where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the identity matrix.

example for finding eigenvalues and eigenvectors

For example, given $X'(t) = AX(t)$ with $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ we have

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

So $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector for the eigenvalue 3. You can get the eigenvalues $\lambda = 3$ and $\lambda = -1$ by solving $det \begin{pmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{pmatrix} = 0.$

This determinant equation is $(1-\lambda)(1-\lambda) - 4 = 0.$

$$\implies \lambda^2 - 2\lambda - 3 = 0 \implies (\lambda - 3)(\lambda + 1) = 0 \implies \lambda = 3 \; or \; -1.$$

Finding the eigenvectors

For each eigenvalue, you need to calculate the eigenvector separately.

For $\lambda = -1$:

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = -1 \begin{pmatrix} u \\ v \end{pmatrix} \implies u+2v = -u, 2u+v = -v \implies u = -v.$$

There is really just one equation and there are infinitely many solutions : one for each value of u, and so the eigenvectors for $\lambda = -1$ are of the form $u \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Similarly all eigenvectors for $\lambda = 3$ are of the form $u \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

General solution using eigenvectors

For each eigenvalue λ we get a solution of the form $e^{\lambda t}$ times the eigenvector.
The general solution is of the form

$$c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t}.$$

Check to confirm that the two eigenvectors are linearly independent (not multiples of each other).
Notice how similar this is to the solution of linear second order equations with constant coefficients. If the auxiliary equation had two roots m_1, m_2 then the general solution was $c_1 e^{m_1 t} + c_2 e^{m_2 t}$.
In the next slide we look at how is it that eigenvectors help in producing the solution.

example for how solutions using eigenvectors work

For this, we see how the eigenvector times e^{3t} is a solution:

$$X(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t} = \begin{pmatrix} 1\,e^{3t} \\ 1\,e^{3t} \end{pmatrix} \implies X'(t) = \begin{pmatrix} 3e^{3t} \\ 3e^{3t} \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t}$$

But from definition of eigenvector for $\lambda = 3$ we have
$3 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Plugging this in,

$$X'(t) = 3 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t}.$$

If we let $X(t) = \begin{pmatrix} 1\,e^{3t} \\ 1\,e^{3t} \end{pmatrix}$ then this equation is saying exactly what the original equation was saying:

$$X'(t) = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} X(t) \text{ !!}$$

Guide for solving systems using eigenvalues

If the eigenvalues are distinct, or atleast the eigenvectors are distinct, then the solution for a system $AX = X'$ is given by $c_1 Ke^{\lambda_1 t} + c_2 Le^{\lambda_2 t}$ where λ_1, λ_2 are the eigenvalues and K, L are the eigenvectors.

Otherwise the solutions are of form $Ke^{\lambda t}$ and $Kte^{\lambda t} + Pe^{\lambda t}$ for some vector P that needs to be solved for using the equation $(A - \lambda I)P = K$.

If there are complex eigenvalues and eigenvectors, then you solve as above and then use the equation $e^{(a+ib)t} = \cos at + i \sin bt$. Note that in this case *you will always get two distinct eigenvalues.* Also, by separating real part and imaginary part, *we get real-values solutions which are independent, so the coefficients in general solution can also be real*.

Very similar to solving 2nd order eqns with auxiliary eqn.

Example 1: system of ODE's

Let us look at a simplified version of the problem from the introduction:
The substance A decays into the substance B which stays stable. The system we get is $x'(t) = -k_1 x(t), y'(t) = k_1 x(t)$.

$$AX = \begin{pmatrix} -k_1 & 0 \\ k_1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix} = X'$$

The characteristic equation is

$$det \begin{pmatrix} -k_1 - \lambda & 0 \\ k_1 & -\lambda \end{pmatrix} = 0.$$

Solving, $(-k_1 - \lambda)(-\lambda) = 0 \implies \lambda = -k_1 \; or \; \lambda = 0$.

Example 1: system of ODE's -page 2

$$AX = \begin{pmatrix} -k_1 & 0 \\ k_1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix} = X'$$

We got the eigenvalues as $\lambda = 0, -k_1$.

Now we find the eigenvectors for these values:

$$\begin{pmatrix} -k_1 & 0 \\ k_1 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = -k_1 \begin{pmatrix} u \\ v \end{pmatrix} \implies \begin{pmatrix} -k_1 u \\ k_1 u \end{pmatrix} = \begin{pmatrix} -k_1 u \\ -k_1 v \end{pmatrix}$$

Comparing both sides in last equation, we see that $-k_1 u = -k_1 u, -k_1 v = k_1 u$.
The first equation tells us nothing. But the second equation says $-k_1 v = k_1 u \implies -v = u$.

Example 1: system of ODE's -page 3

So u can be anything, $v = -u$ means one eigenvector is $\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

All other eigenvectors for $-k_1$ will be multiples of $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and we call $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ as K.

Note that eigenvector for 0, which we will call L, will be different from that of $-k_1$!

Example 1: system of ODE's -page 4

Now we find the eigenvectors for 0:

$$\begin{pmatrix} -k_1 & 0 \\ k_1 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = 0 \begin{pmatrix} u \\ v \end{pmatrix} \implies \begin{pmatrix} -k_1 u \\ k_1 u \end{pmatrix} = 0$$

Comparing both sides in last equation, we see that
$-k_1 u = 0, k_1 u = 0$.
There is no condition on v but the equation says
$k_1 u = 0 \implies u = 0$.
So an eigenvector for 0 is $L = \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

All other eigenvectors for 0 will be multiples of $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Example 1: system of ODE's - page 5

We got, for the eigenvectors,

$$K = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \; L = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The solution for the system is
$c_1 K e^{-k_1 t} + c_2 L e^{0t} = c_1 K e^{-k_1 t} + c_2 L$.
This is very similar to how we get $c_1 e^{-k_1 t} + c_2 e^{0t}$ for the homogenous equation with auxiliary equation $(m + k_1)(m) = 0$. The only difference we get the two vectors $Ke^{k_1 t}$ and Le^{0t}.

Check that this solution agrees with what you get by solving for x from first equation and plugging into second and then integrating second equation.

Example 2: system of ODE's

(Section 8.2.1, problem 1).

$$AX = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix} = X'$$

The characteristic equation is

$$det \begin{pmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{pmatrix} = 0.$$

Solving,
$\lambda^2 - 4\lambda + 3 - 8 = 0 \implies \lambda^2 - 4\lambda - 5 = 0 \implies \lambda = 5, -1.$

contd: Example 2: system of ODE's

(Section 8.2.1, problem 1).

$$AX = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix} = X'$$

We got the eigenvalues as $\lambda = 5, -1$.

Now we find the eigenvectors for these values:

$$\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5x \\ 5y \end{pmatrix} = 5X; \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ -y \end{pmatrix} = -X$$

conclusion: Example 2: system of ODE's

(Section 8.2.1, problem 1).

$$AX = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix} = X'$$

We got, for the eigenvectors,

$$\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5x \\ 5y \end{pmatrix} = 5X; \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ -y \end{pmatrix} = -X$$

The first matrix equation gives $4x - 2y = 0, -4x + 2y = 0$ which is really just $2x - y = 0$. This has infinitely many solutions all satisfying it, in fact it is all the points on a line. Letting $y = 2$, we get $x = 1$. So $K = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is an eigenvector for 5.

Example 2: Final solution

(Section 8.2.1, problem 1). We got, for the eigenvectors,

$$\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5x \\ 5y \end{pmatrix} = 5X; \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ -y \end{pmatrix} = -X$$

The first matrix equation $K = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ as an eigenvector for 5.

Similarly, $L = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ is an eigenvector for -1.

The solution for the system is $c_1 K e^{5t} + c_2 L e^{-t}$.

This is very similar to how we get $c_1 e^{5t} + c_2 e^{-t}$ for the homogenous equation with auxiliary equation $(m-5)(m+1) = 0$. The only difference we get the two vectors Ke^{5t} and Le^{-t}.

Example 3: Repeated eigenvalues

If you have repeated eigenvalues (i.e, the determinant equation only gave one value for λ) then we only get one solution $Ke^{\lambda t}$ where K is the eigenvector for λ.
To get the other solution, we try $X = Kte^{\lambda t}$ just like we did with second order equations when auxiliary equation only had one root.
Unfortunately, this gives $X' = Ke^{\lambda t} + \lambda Kte^{\lambda t}$ which doesn't quite equal $AX = AKte^{\lambda t}$. (we need $X' = AX$).
We do have $AK = \lambda K$ which means $AX = AKte^{\lambda t} = \lambda Kte^{\lambda t}$ but the other term $Ke^{\lambda t}$ is not accounted for.
In next slide we see how to get around this.

Example 2: Repeated eigenvalues – page 2

The problem is fixed if you try $X = Kte^{\lambda t} + Pe^{\lambda t}$ where P is an unknown, but fixed vector, that can be figured out as follows: With X as above, $X' = Ke^{\lambda t} + \lambda Kte^{\lambda t} + P\lambda e^{\lambda t}$ and this needs to equal $AX = AKte^{\lambda t} + APe^{\lambda t}$. As shown in previous slide, $AKte^{\lambda t} = \lambda Kte^{\lambda t}$ so we need

$$Ke^{\lambda t} + P\lambda e^{\lambda t} = APe^{\lambda t}. \quad (1)$$

We will use equation (1) to find the solutions for problem 21 in 8.2.

Example 2: Repeated eigenvalues – problem 21, 8.2

For the problem 21, $A = \begin{pmatrix} 3 & -1 \\ 9 & -3 \end{pmatrix}$. As before, we find the eigenvalues λ using the equation $det(A - \lambda I) = 0$ and get $det\left(\begin{pmatrix} 3-\lambda & -1 \\ 9 & -3-\lambda \end{pmatrix}\right) = 0$ which gives $\lambda = 0$. The eigenvector K is got by setting $AK = 0K = 0$ and we get two equations $3k_1 - k_2 = 0, 9k_1 - 3k_2 = 0$ which are really the same. The eigenvectors for $\lambda = 0$ are multiples of $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$.

So one solution is $\begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{\lambda t} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$.

To find the other solution we use equation (1): (next slide)

Example 2: Repeated eigenvalues – problem 21, 8.2, page 2

Put $\lambda = 0, K = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $A = \begin{pmatrix} 3 & -1 \\ 9 & -3 \end{pmatrix}$ in equation (1) (with $P = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$) we get:

$$\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ 9 & -3 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}. \quad (2)$$

From equation (2) we get two equations which are really just multiples of the one equation $3p_1 - p_2 = 1$. This has parametric solutions $P = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} p_1 \\ 3p_1 - 1 \end{pmatrix}$

Each value of p_1 gives one vector. For example, $p_1 = 1$ gives $p_2 = 2$. (In book $p_1 = 1/4, p_2 = -1/4$ is given). (Next slide)

Example 2: Repeated eigenvalues – problem 21, 8.2, page 3

Remember that we got the solution $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ as the eigenvector for $\lambda = 0$.

The other solution was supposed to be
$\begin{pmatrix} 1 \\ 3 \end{pmatrix} te^{\lambda t} + P = \begin{pmatrix} 1 \\ 3 \end{pmatrix} t + P$.

We got $P = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

So putting everything together, the general solution is

$$X(t) = c_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} + c_2 \left(\begin{pmatrix} 1 \\ 3 \end{pmatrix} t + \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right).$$

Example 2: system of ODE's

(Section 8.2.1, problem 1).

$$AX = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix} = X'$$

The characteristic equation is

$$det \begin{pmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{pmatrix} = 0.$$

Solving,
$\lambda^2 - 4\lambda + 3 - 8 = 0 \implies \lambda^2 - 4\lambda - 5 = 0 \implies \lambda = 5, -1.$

contd: Example 2: system of ODE's

(Section 8.2.1, problem 1).

$$AX = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix} = X'$$

We got the eigenvalues as $\lambda = 5, -1$.

Now we find the eigenvectors for these values:

$$\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5x \\ 5y \end{pmatrix} = 5X; \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ -y \end{pmatrix} = -X$$

conclusion: Example 2: system of ODE's

(Section 8.2.1, problem 1).

$$AX = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix} = X'$$

We got, for the eigenvectors,

$$\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5x \\ 5y \end{pmatrix} = 5X; \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ -y \end{pmatrix} = -X$$

The first matrix equation gives $4x - 2y = 0, -4x + 2y = 0$ which is really just $2x - y = 0$. This has infinitely many solutions all satisfying it, in fact it is all the points on a line. Letting $y = 2$, we get $x = 1$. So $K = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is an eigenvector for 5.

Example 2: Final solution

(Section 8.2.1, problem 1). We got, for the eigenvectors,

$$\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5x \\ 5y \end{pmatrix} = 5X; \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ -y \end{pmatrix} = -X$$

The first matrix equation $K = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ as an eigenvector for 5.

Similarly, $L = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ is an eigenvector for -1.

The solution for the system is $c_1 K e^{5t} + c_2 L e^{-t}$.

This is very similar to how we get $c_1 e^{5t} + c_2 e^{-t}$ for the homogenous equation with auxiliary equation $(m-5)(m+1) = 0$. The only difference we get the two vectors Ke^{5t} and Le^{-t}.

Example 3: Complex Eigenvalues

$$AX = \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix} = X'$$

The characteristic equation is

$$det \begin{pmatrix} 1-\lambda & -1 \\ 2 & -1-\lambda \end{pmatrix} = 0.$$

Solving, $\lambda^2 - 1 + 2 = 0 \implies \lambda^2 + 1 = 0 \implies \lambda = i, -i.$

contd: Example 3: complex eigenvalues

$$AX = \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix} = X'$$

We got the eigenvalues as $\lambda = i, -i$.

Now we find the eigenvectors for these values:

$$\begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} ik_1 \\ ik_2 \end{pmatrix} ; \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} -ik_1 \\ -ik_2 \end{pmatrix}$$

conclusion: Example 3: complex eigenvalues

We got, for the eigenvectors,

$$\begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} ik_1 \\ ik_2 \end{pmatrix}; \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} -ik_1 \\ -ik_2 \end{pmatrix}.$$

The first matrix equation gives $(1-i)k_1 - k_2 = 0, 2k_1 - (1+i)k_2 = 0$ which is really just $(1-i)k_1 - k_2 = 0$ because second equation is $(1+i)$ times first. This has infinitely many solutions all satisfying it, in fact it is all the points on a line. Letting $k_1 = 1$, we get $k_2 = 1 - i$. So $K = \begin{pmatrix} 1 \\ 1-i \end{pmatrix}$ is an eigenvector for i.

Similarly, $L = \begin{pmatrix} 1 \\ 1+i \end{pmatrix}$ is an eigenvector for $-i$.

Example 3: complex eigenvalues – complex general solution

We got, for the eigenvectors, $K = \begin{pmatrix} 1 \\ 1 - i \end{pmatrix}$ is an eigenvector for i, $L = \begin{pmatrix} 1 \\ 1 + i \end{pmatrix}$ is an eigenvector for $-i$.
[*Notice how the eigenvalues and eigenvectors are conjugates of each other – this is a feature of complex numbers and complex functions*.]

The solution for the system is $c_1 K e^{it} + c_2 L e^{-it}$.

We really want *real valued functions*, so we **separate the real and imaginary parts** and get two independent solutions which are just real valued functions.

Example 3: complex eigenvalues –real solutions

So we need to separate the real and imaginary parts from $c_1 K e^{it} + c_2 L e^{-it}$ and get two independent solutions which are just real valued functions.

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 K e^{it} + c_2 L e^{-it} = c_1 \begin{pmatrix} 1 \\ 1-i \end{pmatrix} e^{it} + c_2 \begin{pmatrix} 1 \\ 1+i \end{pmatrix} e^{-it}.$$

$$\implies x(t) = c_1 e^{it} + c_2 e^{-it}, y(t) = c_1(1-i)e^{it} + c_2(1+i)e^{-it}.$$

Now *separate real and imaginary parts:* (next slide)

Example 3: complex eigenvalues –real solutions -page 2

We had $x(t) = c_1 e^{it} + c_2 e^{-it}, y(t) = c_1(1-i)e^{it} + c_2(1+i)e^{-it}$.
Using $e^{i\alpha t} = \cos(\alpha t) + i \sin \alpha t$, we have
$e^{it} = \cos t + i \sin t, e^{-it} = \cos t - i \sin t$.
Plug this into the equation for $x(t), y(t)$ from the above equation.
$x(t) = c_1(\cos t + i \sin t) + c_2(\cos t - i \sin t) =$
$(c_1 + c_2) \cos t + i(c_1 - c_2) \sin t$.
$y(t) = c_1(1-i)(\cos t + i \sin t) + c_2(1+i)(\cos t - i \sin t) =$
$(c_1 + c_2)(\cos t + \sin t) + i(c_1 - c_2)(\sin t - \cos t)$.

$$So \quad \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = (c_1 + c_2) \begin{pmatrix} \cos t \\ \cos t + \sin t \end{pmatrix} + i(c_1 - c_2) \begin{pmatrix} \sin t \\ \sin t - \cos t \end{pmatrix}$$

Example 3: complex eigenvalues –real solutions - conclusion

So $\begin{pmatrix} \cos t \\ \cos t + \sin t \end{pmatrix}$ and $\begin{pmatrix} \sin t \\ \sin t - \cos t \end{pmatrix}$, the real and imaginary parts of the complex solution, are solutions of original equation $X' = AX$. Check it for yourself!
The nice thing is that now we have a general solution in terms of real valued functions only.

But why are real and imaginary parts also solutions? Reason:

We operate on real and imaginary parts separately, whether it is differentiation or integration.
Example: $f(t) = u(t) + i(v(t) \implies f'(t) = u'(t) + iv'(t)$. So if $f(t)$ satisfies $f'(t) = 2f(t)$, then
$u' + iv' = 2(u + iv) \implies u' = 2u, v' = 2v$. You see that the real and imaginary parts *also satisfy the same equation!*

Note about complex eigenvalues and solutions

The following is just for your information. It needs a bit of understanding of how complex numbers, matrices and functions behave. You can use it as a short-cut, if you like.

Conjugates of eigenvalues are also eigenvalues. Denoting conjugate of a complex number α by $\overline{\alpha}$, we have
$\alpha = a + ib \implies \overline{\alpha} = a - ib$.
If $AK = \lambda K$, then taking conjugates $\overline{AK} = \overline{A}\,\overline{K} = \overline{\lambda}\,\overline{K}$.
But if A were a real matrix, then $\overline{A} = A$, so $= \overline{A}\,\overline{K} = A\overline{K} = \overline{\lambda}\,\overline{K}$.
So $\overline{\lambda}$ *is another eigenvalue with eigenvector* $\overline{K}$.

For instance, if $\begin{pmatrix} 1+i \\ 1-i \end{pmatrix}$ were an eigenvector for $\lambda = 2 + i$, then $\begin{pmatrix} 1-i \\ 1+i \end{pmatrix}$ would be an eigenvector for $\overline{\lambda} = 2 - i$.

Note about complex eigenvalues and solutions – contd

Continued from previous slide:

You can also use the ideas in previous slide to simplify calculation of real part and imaginary part in the general solution. For example, if you had $X = c_1 Ke^{it} + c_2 \overline{K} e^{-it}$ then let $Ke^{it} = A + iB$. Then $\overline{K} e^{-it} = \overline{A + iB} = A - iB$.

$$X = c_1 Ke^{it} + c_2 \overline{K} e^{-it} = c_1(A + iB) + c_2(A - iB) = (c_1 + c_2)A + i(c_1 - c_2)B.$$

In other words, *we only need to find the real part A and imaginary part B of one solution* and the combined solution will have real part $(c_1 + c_2)A$ and imaginary part $(c_1 - c_2)B$. Mainly because the two solutions are conjugates of each other.

Note about complex eigenvalues and solutions – conclusion

For instance, $Re\left(c_1\begin{pmatrix}1\\1-i\end{pmatrix}e^{it}+c_2\begin{pmatrix}1\\1+i\end{pmatrix}e^{-it}\right)$ equals $(c_1+c_2)Re\left(\begin{pmatrix}1\\1-i\end{pmatrix}e^{it}\right)=(c_1+c_2)\begin{pmatrix}\cos t\\\cos t+\sin t\end{pmatrix}$.

Similarly, $Im\left(c_1\begin{pmatrix}1\\1-i\end{pmatrix}e^{it}+c_2\begin{pmatrix}1\\1+i\end{pmatrix}e^{-it}\right)$ equals $(c_1-c_2)Im\left(\begin{pmatrix}1\\1-i\end{pmatrix}e^{it}\right)=(c_1-c_2)\begin{pmatrix}\sin t\\\sin t-\cos t\end{pmatrix}$.

Example 4: Systems solution using Laplace transforms

We can solve systems like the one we just solved, $AX = \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix} = X'$, using Laplace transform as well!

Let $x(0) = y(0) = 1$. Start by applying Laplace transform to both sides of each aquation:

$$\begin{matrix} x' = x - y \\ y' = 2x - y \end{matrix} \implies \begin{matrix} L(x') = L(x - y) \\ L(y') = L(2x - y) \end{matrix}$$

$$\implies \begin{matrix} sL(x) - x(0) = L(x) - L(y) \\ sL(y) - y(0) = 2L(x) - L(y) \end{matrix}$$

$$\implies \begin{matrix} (s-1)L(x) + L(y) = 1 \\ -2L(x) + (s+1)L(y) = 1 \end{matrix}$$

Example 4: Systems solution using Laplace transforms – page 2

$$(s-1)L(x) + L(y) = 1$$
$$-2L(x) + (s+1)L(y) = 1$$

Solve this system by multiplying first equation by -2 and second by $s-1$ and subtracting. Get
$((s^2-1)+2)L(y) = (s-1)+2 \implies L(y) = (s+1)/(s^2+1)$.
So $y(t) = L^{-1}\left(\frac{s+1}{s^2+1}\right) = L^{-1}\left(\frac{s}{s^2+1}\right) + L^{-1}\left(\frac{1}{s^2+1}\right) = \cos t + \sin t$.
Then
$L(x) = (1-L(y))/(s-1) = (s^2-s)/((s^2+1)(s-1)) = s/(s^2+1)$
and $x(t) = L^{-1}(s/(s^2+1)) = \cos t$. This is the same as the first solution we had before.
For $x(0) = 0, y(0) = -1$ we get $x(t) = \sin t$, $y(t) = \sin t - \cos t$, which is the same as the second solution we had before.