Differential Equations Final Exam Solutions
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Howard University Mathematics Department

MUST GIVE STEP BY STEP EXPLANATIONS TO GET CREDIT FOR ANSWERS.
No calculators or electronic devices are permitted.

PART I

Do ALL 3 problems. EACH WORTH 20 POINTS.

1. Find the general solution of the following linear equation. Give the largest interval over which the general solution is defined.

$$\frac{dy}{dx} - y = xe^x.$$

   Solution:

   This is of form $y' + P(x)y = Q(x)$.

   The integrating factor $e^{\int P dx} = e^{\int -dx} = e^{-x}$.

   Multiplying by integrating factor and integrating, we get

$$d(ye^{-x}) = e^{-x}(xe^x) \implies ye^{-x} = \int xdx + C \implies ye^{-x} = \frac{x^2}{2} + C.$$

   So the solution is $y = x^2e^x/2 + Ce^x$.

   The solution is valid in $(-\infty, \infty)$ because the coefficient of y', in this case 1, is never zero.

2. For the ODE $y'' - 2y' + y = 0$ find the general solution, and then find the solution satisfying $y(0) = 1, y'(0) = 0$.

   Solution:

   The auxiliary equation is $m^2 - 2m + 1 = 0$ and it has only one root $m = 1$ using factoring or quadratic formula.

   So the general solution is $y = ae^x + bxe^x$.

   For initial value $y(0) = 1$ we get $1 = a$.

   For initial value $y'(0) = 0$ we get $e^x + be^x + bxe^x = y' \implies 1 + b = 0 \implies b = -1$ and so the required solution is $e^x - xe^x = e^x(1 - x)$.

3. Let $x(t)$ be the position of a particle at time t. Then $x'(t)$ is the velocity, $x''(t)$ the acceleration at time t. It is given that the acceleration at any time t is equal to 32 $feet/sec^2$.

   (a) Write down the differential equation of motion for this particle.

   (b) Why is the equation non-homogenous? What is the homogenous equation?

(c) Solve the differential equation and get the general solution. You can use any method you like.

Solution: The equation is $x''(t) = 32$. It is non-homogenous because RHS is not 0. Integrating, we get

$$x''(t) = 32 \implies x'(t) = 32t + A \implies x(t) = 32(t^2/2) + At + B = 16t^2 + At + B.$$

The homogenous part is $x''(t) = 0$.

We can also do this using the auxiliary equation $m^2 = 0$ and it has only one root $m = 0$.

So the general solution of homogenous part is $y = ae^{0x} + bte^{0t} = a + bt$.

NOTE: Many of you confused homogenous expression (such as $x^2 + y^2 + xy$) with homogenous differential equation. We do have ODE's with homogenous expressions but they are not same as homogenous ODE's.

To get the particular solution for $x''(t) = 32$ we use $x(t) = ct^2$ (don't need $a+bt+ct^2$ because t and 1 are already in the complimentary solution).

We get $x''(t) = 2c \implies 2c = 32 \implies c = 16$.

So the solution is $x(t) = x_c + x_p = a + bt + 16t^2$.

PART II

Choose any 7 problems. EACH WORTH 20 POINTS.

1. Solve the separable ODE $x^2y' = y - xy$ with initial value $y(-1) = 4$.

   Solution: We get $dy/y = (1-x)dx/x^2$ by simplifying. Integrating, we get $\ln|y| = (-1/x) - \ln|x| + C$. This means

$$\ln|y| + \ln|x| = C - (1/x) \implies |yx| = e^{C-\frac{1}{x}} = e^C e^{-1/x}.$$

   Now using the initial value, we get $-4 = \pm e^C \times e \implies e^C = \pm 4/e$. But e^C is always positive, so it must equal $4/e$. So the solution is

$$yx = \pm\frac{4e^{-\frac{1}{x}}}{e} \implies y = \pm 4\frac{e^{-\frac{1}{x}-1}}{x}.$$

   We need to choose the negative sign so as to get $y(-1) = 4$. So the final solution is

$$y = -4\frac{e^{-\frac{1}{x}-1}}{x}.$$

2. The number of infected people $P(t)$ changes at a rate proportional to $P(t)$ itself and the people not infected.

   (a) If total population was 1000, write down a differential equation that models this problem.

(b) Using phase portrait, show what happens to $P(t)$ as $t \to \infty$. You don't need to solve for $P(t)$.

Solution: We get $dP/dt = kP(1000 - P)$. The critical points are 0 and 1000. Above 1000, $P(1000 - P)$ is negitive, so $dP/dt < 0$ and P is decreasing. Below 1000 it is positive and P is increasing. So P would approach 1000 from both above and below as $t \to \infty$.

3. Use Euler's method to obtain a four-decimal approximation of $y(0.3)$ with $h = 0.1$ from the IVP: $y' = xy,\ y(0) = 1$.

   Solution: $f(x, y) = xy$. Using Euler's method, $y(0.1) = y(0)+hf(0, 1) = 1+(0.1)(0) = 1$; $y(0.2) = y(0.1) + hf(0.1, 1) = 1 + 0.1(0.1) = 1.01$; $y(0.3) = y(0.2) + hf(0.2, 1.01) = 1.01 + 0.1(0.2 \times 1.01) = 1.0302$.

4. Solve using substitution method:

$$\frac{dy}{dx} = (x + y + 1)^2.$$

   Solution:

   Actually a non-linear, non-exact, non-separable equation.

   The substitution that works here is $u = x + y + 1$.

$$u = x + y + 1 \implies \frac{du}{dx} = 1 + \frac{dy}{dx}$$

$$\frac{dy}{dx} = (x + y + 1)^2 \implies \frac{du}{dx} - 1 = u^2$$

   This is a separable equation if you write as $u' = u^2 + 1$ and can be solved.

   You get $\tan^{-1} u = x+C \implies u = x+y+1 = \tan(x+C) \implies y = \tan(x+C)-x-1$.

5. Show that $y'' - y = 0$ has fundamental solutions e^x, e^{-x}. Then show linear independence using Wronskian or by showing that $ae^x + be^{-x} = 0$ for all x means $a = 0, b = 0$.

   Solution:

   The auxiliary equation is $m^2 - 1 = 0$ and it has roots $1, -1$. So we get the solutions e^x, e^{-x}.

   You can also just show that they satisfy the given equation.

   Wronskian is $e^x(e^{-x})' - (e^x)'e^{-x} = -1 - 1 = -2$ which is not zero.

   If $ae^x + be^{-x} = 0$ for all x then we can put $x = 0, 1$ and solve to get $b = 0, a = 0$.

6. Mass weighing 24 lbs gets a spring stretched 4 inches. It is released 3 inches above equilibrium position with downward velocity of 2 ft/s. Write the differential equation for this system and solve it to get the equation of motion. [Note: 1 foot equals 12 inches].

   Solution: 4 inches is 1/3 foot. From Hooke's law, $24 = k(1/3) \implies k = 72$.

$m = 24/32 = 3/4 = 0.75$ slugs. (weight is the force due to gravity and it equals mg where m is mass and g is the acceleration due to gravity which is 32 ft/sec/sec).

$\omega^2 = k/m = 72/(3/4) = 96 \implies \omega = 4\sqrt{6}$.

We found $\omega = 4\sqrt{6}$. So solution is $x(t) = c_1 \cos(4\sqrt{6}t) + c_2 \sin(4\sqrt{6}t)$.

Now we use the initial conditions: it is released 3 inches (1/4 foot) above equilibrium position with downward velocity of 2 ft/s.

So $x(0) = -1/4, x'(0) = 2$.

This gives $c_1 = -1/4, c_2 = 1/(2\sqrt{6})$. So the solution is $x(t) = -(1/4)cos(4\sqrt{6}t) + (1/(2\sqrt{6}))\sin(4\sqrt{6}t)$.

7. Using power series find two independent solutions for $y'' + xy = 0$.

Solution: $y'' + xy = 0$ has no singular points, the coefficient of y'' is 1 which is defined everywhere.

To solve $y'' + xy = 0$ with $y = \sum_{n=0}^{\infty} a_n x^n$, $y'' = \sum_{n=0}^{\infty} a_n(n(n-1))x^{n-2}$ we jut plug them into equation first. Then we look at coefficient of x^n everywhere.

$$\sum_{n=0}^{\infty} a_n(n(n-1))x^{n-2} + x\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\implies a_{n+2}(n+2)(n+1) + a_{n-1} = 0.$$

This is the recurrence relation.

We start with $n = 1$ because $n = 0$ gives a_{-1}.

Now $a_2 = 0$ because there is no constant term in $x\sum_{n=0}^{\infty} a_n x^n$ and the coefficient of the constant term in $\sum_{n=0}^{\infty} a_n(n(n-1))x^{n-2}$ is $2a_2$ and it has to equal zero because the whole thing equals zero and so coefficients are all equal to 0 (that is how we get the recurrence relation $a_{n+2}(n+2)(n+1) + a_{n-1} = 0$).

Using the recurrence relation, we get

$$n = 1 \implies a_3 = -\frac{a_0}{2\cdot 3},\ n = 2 \implies a_4 = -\frac{a_1}{3\cdot 4}$$

$$n = 3 \implies a_5 = -\frac{a_2}{4\cdot 5} = 0,\ n = 4 \implies a_6 = -\frac{a_3}{5\cdot 6} = -(-\frac{a_0}{2\cdot 3})/30 = \frac{a_0}{180}$$

$$y_1 = 1 - \frac{x^3}{6} + \frac{x^6}{180}.....,\ y_2 = x - \frac{x^4}{12} + \frac{x^7}{504}....$$

Basically all coefficients are either multiples of a_0 or of a_1. Separating the two constants, we get the general solution $y = a_0 y_1 + a_1 y_2$ where y_1, y_2 are two power series.

The two power series are

8. Solve $y' - 2y = e^t, y(0) = 0$ using Laplace transforms.

Solution:

To solve $y' - 2y = e^t, y(0) = 0$ we use Laplace transform and the formula $L(f'(t)) = -f(0) + sL(f)$.

We have $L(f(t)) = L(y) = F(s), y(0) = f(0) = 0$.

Start by taking Laplace Transform of both sides: $L(y' - 2y) = L(e^t)$. Using linearity $L(y' - 2y) = L(y') - 2L(y)$.

Now using formula for $L(y')$ we get

$$sF(s) - f(0) - 2F(s) = \frac{1}{s-1} \implies F(s) = \frac{1}{(s-2)(s-1)} = \frac{1}{s-2} - \frac{1}{s-1}$$

$$\implies f(t) = L^{-1}F(s) = L^{-1}(1/(s-2)) - L^{-1}(1/(s-1)) = e^{2t} - e^t.$$

9. Solve the following system:

$$x'(t) = x + 2y; y'(t) = 4x + 3y.$$

Solution: First we write in matrix form:

$$AX = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix} = X'$$

The characteristic equation is

$$det \begin{pmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{pmatrix} = 0.$$

Solving, $\lambda^2 - 4\lambda + 3 - 8 = 0 \implies \lambda^2 - 4\lambda - 5 = 0 \implies \lambda = 5, -1$.

$$AX = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix} = X'$$

We got the eigenvalues as $\lambda = 5, -1$.

Now we find the eigenvectors for these values:

$$\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5x \\ 5y \end{pmatrix} = 5X; \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ -y \end{pmatrix} = -X$$

$$AX = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix} = X'$$

We got, for the eigenvectors,

$$\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5x \\ 5y \end{pmatrix} = 5X; \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ -y \end{pmatrix} = -X$$

The first matrix equation gives $4x - 2y = 0, -4x + 2y = 0$ which is really just $2x - y = 0$. This has infinitely many solutions all satisfying it, in fact it is all the points on a line. Letting $y = 2$, we get $x = 1$. So $K = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is an eigenvector for 5.

Final solution :

Similarly, $L = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ is an eigenvector for -1.

The solution for the system is $c_1 K e^{5t} + c_2 L e^{-t}$.