Calculus I Final Exam Solution : Version 1
December 1, 2020
Howard University Mathematics Department

MUST GIVE STEP BY STEP EXPLANATIONS TO GET CREDIT FOR ANSWERS.
No calculators or electronic devices are permitted.

PART I

Do all three problems. EACH WORTH 24 POINTS.

1. For the function $f(x) = x^4 - 4x^3 + 10$,

   (a) Find the intervals on the x-axis where it is increasing, decreasing.

   (b) Find the intervals on the x-axis where it is concave up and where it is concave down.

   (c) Find the inflexion points.

   (d) Using the information from (i) to (iii) along with its behavior at $\pm\infty$ to graph the function.

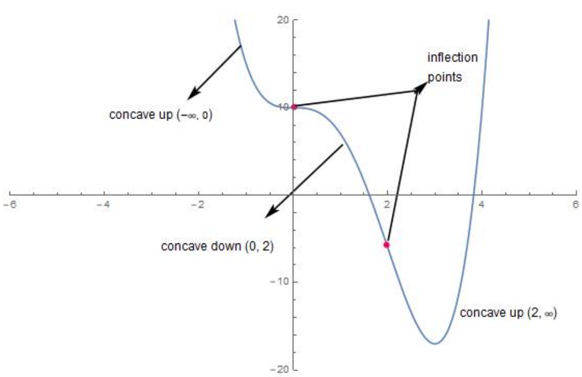
   Solution: Given $f(x) = x^4 - 4x^3 + 10 \Rightarrow f'(x) = 4x^3 - 12x^2$.
   $f'(x) = 0 \Rightarrow 4x^2(x-3) = 0 \Rightarrow x = 0$ *and* $x = 3$. Hence 0 and 3 are the critical numbers.

   (a) $(-\infty, 0)$: function is decreasing ; $(0, 3)$: function is decreasing ; $(3, \infty)$: function is increasing.

   (b) We have $f''(x) = 12x^2 - 24x$. $f''(x) = 0 \Rightarrow 12x(x-2) = 0 \Rightarrow x = 0, x = 2$.
   $(-\infty, 0)$: function is concave up ; $(0, 2)$: function is concave down ; $(2, \infty)$: function is concave up.

   (c) Inflection points are : $(0, 10)$ & $(2, -6)$.

   (d) $\lim_{x\to\pm\infty} f(x) = \infty$.



2. For the function $f(x) = \sqrt{\dfrac{x+1}{1-x^3}}$

   (a) Find the derivative $f'(x)$.

(b) Find an equation of the tangent line to the curve at $x = 0$.

Solution: Given $f(x) = \sqrt{\dfrac{x+1}{1-x^3}}$.

(a)

$$\begin{aligned}
f(x) &= \left(\frac{x+1}{1-x^3}\right)^{\frac{1}{2}} \\
f'(x) &= \frac{1}{2}\left(\frac{x+1}{1-x^3}\right)^{-\frac{1}{2}} \frac{d}{dx}\left(\frac{x+1}{1-x^3}\right) \\
f'(x) &= \frac{1}{2}\left(\frac{x+1}{1-x^3}\right)^{-\frac{1}{2}} \left[\frac{(1-x^3).1-(x+1).(-3x^2)}{(1-x^3)^2}\right] \\
f'(x) &= \frac{1}{2}\left(\frac{x+1}{1-x^3}\right)^{-\frac{1}{2}} \left[\frac{1-x^3+3x^3+3x^2}{(1-x^3)^2}\right] \\
f'(x) &= \frac{2x^3+3x^2+1}{2\sqrt{\left(\frac{x+1}{1-x^3}\right)}(1-x^3)^2}
\end{aligned}$$

(b) At $x = 0$, $f(0) = 1$, $f'(0) = \frac{1}{2}$
The equation of the tangent line is : $(y-1) = \frac{1}{2}(x-0) \Rightarrow y = \frac{1}{2}x + 1$

3. The height of a ball thrown up, after t seconds, is given by $h(t) = 4t - t^2 + 12$ feet. Find the following:

(a) Initial position and velocity after 1 second.

(b) Acceleration after 5 seconds.

(c) Maximum height.

(d) Time when it is comes back down.

Solution: Given $h(t) = -t^2 + 4t + 12$.

(a) Initial position : $h(0) = 12$ feet.
Velocity : $v(t) = h'(t) = -2t + 4$ After 1 second, velocity is : $v(1) = 2$ ft/s.

(b) Acceleration : $a(t) = v'(t) = -2$ ft/s.

(c) To find maximum height, we need to find time taken to reach maximum height. The maximum height is the point where the ball changes direction and at this point the derivative will be 0. $h'(t) = -2t + 4 = 0 \Rightarrow t = 2$ second. The ball takes 2 seconds to reach maximum height. Hence the maximum height will be $h(2) = 16$ feet.

(d) When the ball comes back down, height will be 0. i.e., $-t^2 + 4t + 12 = 0 \Rightarrow (t-6)(t+2) = 0$. The time when the ball comes back down is 6 seconds.

PART II

Choose any 8 problems. EACH WORTH 16 POINTS.

1. Given

$$f(x) = \begin{cases} x^2+1 & if\ x<1 \\ (x-2)^2 & if\ x \geq 1 \end{cases}$$

   (a) Find $\lim_{x\to1^-} f(x)$ and $\lim_{x\to1^+} f(x)$

   (b) Does $\lim_{x\to1} f(x)$ exist?

   (c) Sketch the graph of f and justify your answer.

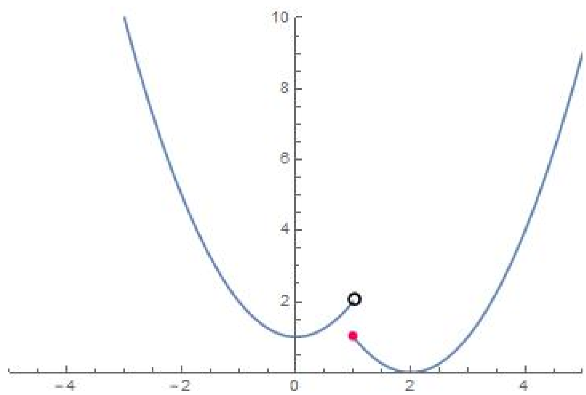
   Solution: Given

$$f(x) = \begin{cases} x^2+1 & if\ x<1 \\ (x-2)^2 & if\ x \geq 1 \end{cases}$$

   (a) $\lim_{x\to1^-} f(x) = 2$ and $\lim_{x\to1^+} f(x) = 1$

   (b) $\lim_{x\to1^-} f(x) \neq \lim_{x\to1+} f(x)$. Hence the limit does not exist.

   (c) The graph of the given function is:



   From the graph, it is clear that function is discontinuous at $x = 1$.

2. The doubling time of an investment of \$ 10000 under continuous compounding is 10 years. Write the equation for the amount at time t years using an exponential function to the base 2 (answer involves a power of 2). Then write it as a natural exponential function, i.e, to the base e.

   Solution: Doubling time model for the given problem is : $A(t) = 1000\left(2^{\frac{t}{10}}\right)$

3. Let $f(x) = \dfrac{x^3}{4} + 1$. Check that it satisfies the conditions of the Mean Value Theorem on [0,2]. Find the value or values of c in $(0,2)$ for which the tangent at $(c, f(c))$ is parallel to secant from $(0,1)$ to $(2,3)$.

   Solution: Given $f(x) = \dfrac{x^3}{4} + 1$

   (a) The function is continuous on $[0,2]$.

   (b) The function is differentiable on $(0,2)$.

   (c) Also, $f(0) \neq f(2)$

By Mean Value Theorem, there exists, a number c such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

$$\begin{aligned} \frac{3}{4}c^2 &= \frac{f(2) - f(0)}{2} \\ c^2 &= \frac{4}{3}\left(\frac{3-1}{2}\right) \\ c &= \pm\frac{2}{\sqrt{3}} \end{aligned}$$

Therefore, the tangent at $\left(\dfrac{2}{\sqrt{3}}, 1 + \dfrac{2}{3\sqrt{3}}\right)$ is parallel to secant line on $[0, 2]$.

4. Compute the derivatives of the following functions:
   (a) $f(x) = \dfrac{(x^3 - 2x^4)(3 - 5x)}{x^3 + 3x + 1}$ (use logarithmic differentiation).
   (b) $y = x^2 \ln(x + 2) + 2^x$

   Solution:

   (a) Given $f(x) = \dfrac{(x^3 - 2x^4)(3 - 5x)}{x^3 + 3x + 1}$

   Taking natural log on both sides,

$$\begin{aligned} \ln f(x) &= \ln\left(\frac{(x^3 - 2x^4)(3 - 5x)}{x^3 + 3x + 1}\right) \\ \ln f(x) &= \ln\left((x^3 - 2x^4)(3 - 5x)\right) - \ln\left(x^3 + 3x + 1\right) \\ \ln f(x) &= \ln(x^3 - 2x^4) + \ln(3 - 5x) - \ln\left(x^3 + 3x + 1\right) \\ \frac{1}{f}\frac{df}{dx} &= \frac{1}{x^3 - 2x^4}(3x^2 - 8x^3) + \frac{1}{3 - 5x}(-5) - \frac{1}{x^3 + 3x + 1}(3x^2 + 3) \\ \frac{df}{dx} &= f\left(\frac{3x^2 - 8x^3}{x^3 - 2x^4} - \frac{5}{3 - 5x} - \frac{3(x^2 + 1)}{x^3 + 3x + 1}\right) \end{aligned}$$

   (b) Given $y = x^2 \ln(x + 2) + 2^x$

$$\begin{aligned} y' &= x^2\frac{d}{dx}(\ln(x + 2)) + \ln(x + 2)\frac{d}{dx}(x^2) + \frac{d}{dx}(2^x) \\ y' &= \frac{x^2}{x + 2} + 2x\ln(x + 2) + 2^x\ln(2) \end{aligned}$$

5. Use implicit differentiation to find the derivative of $\sin^{-1} x$.
   Remember that $y = \sin^{-1} x$ means $x = \sin y$ in the domain $-\dfrac{\pi}{2} \le y \le \dfrac{\pi}{2}$.

   Solution: Let $y = \sin^{-1}(x) \Rightarrow x = \sin y$.
   Using implicit differentiation,

$$\cos y\frac{dy}{dx} = 1$$

$$\begin{aligned}
\frac{dy}{dx} &= \frac{1}{\cos y} \\
\frac{dy}{dx} &= \frac{1}{\sqrt{1-\sin^2 y}} \quad (\sin^2 y + \cos^2 y = 1) \\
\frac{d}{dx}(\sin^{-1} x) &= \frac{1}{\sqrt{1-x^2}}
\end{aligned}$$

6. Find the linearization (linear approximation) $L(x)$ of $f(x) = \sqrt[3]{x}$ at $a = 1$ and use it to approximate $\sqrt[3]{1.03}$.

   Solution: Given $f(x) = \sqrt[3]{x} \Rightarrow f'(x) = \frac{1}{3x^{\frac{2}{3}}}$. We have $L(x) = f(a) + f'(a)(x-a)$. At $a = 1$,

$$\begin{aligned}
L(x) &= f(1) + f'(1)(x-1) \\
L(x) &= 1 + \frac{1}{3}(x-1)
\end{aligned}$$

   To approximate $\sqrt[3]{1.03}$, substitute $x = 1.03$ in $L(x)$:
   i.e., $L(1.03) = 1 + \frac{1}{3}(1.03-1) = 1.01$

7. Evaluate the following limits and justify each step by indicating appropriate limit properties:

   (a) $\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x}$ (without L'Hospital's Rule)

   (b) $\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x}$ (with L'Hospital's Rule)

   Solution:

   (a) Given $\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x}$

$$\begin{aligned}
\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x} &= \lim_{x\to 0} \left(\frac{\sqrt{x+1}-1}{x}\right)\left(\frac{\sqrt{x+1}+1}{\sqrt{x+1}+1}\right) \\
&= \lim_{x\to 0} \left(\frac{x+1-1}{x(\sqrt{x+1}+1)}\right) \\
&= \lim_{x\to 0} \left(\frac{1}{(\sqrt{x+1}+1)}\right) \\
&= \frac{1}{2}
\end{aligned}$$

   (b) Given $\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x}$. By L'Hospital's rule,

$$\begin{aligned}
\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x} &= \lim_{x\to 0} \frac{\frac{1}{2\sqrt{x+1}}}{1} \\
\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x} &= \frac{1}{2}
\end{aligned}$$

8. A closed cylindrical can needs to hold 1000 cubic centimeters of liquid. Find the height and radius so that the material needed to make the can (surface area) is minimized. Your answer will show that the optimum height equals twice the radius.

   Solution : Let h be the height and r be the radius of the base.
   Given, $Volume = \pi r^2 h = 1000cc$. We need to find height and radius for which the surface area will be minimum.
   We have Surface area of the can, i.e.,

$$
\begin{aligned}
S &= 2\pi r^2 + 2\pi r h \\
S(r) &= 2\pi \left(r^2 + r\frac{1000}{\pi r^2} \right) \\
&= 2\pi \left(r^2 + \frac{1000}{\pi r} \right) \\
S' &= 2\pi \left[2r - \frac{1000}{\pi r^2} \right]
\end{aligned}
$$

$$S' = 0 \Rightarrow \left[2r - \frac{1000}{\pi r^2} \right] = 0 \quad \Rightarrow \quad 2\pi r^3 = 1000 \quad \Rightarrow \quad r = \sqrt[3]{\frac{500}{\pi}} \approx 5.42$$

   Here $S'(r) < 0$ for all $r < 0$ and $S'(r) > 0$ for all $r > 0$. Hence for $r = \sqrt[3]{\frac{500}{\pi}}$, surface area will be minimum. The corresponding height will be

$$
\begin{aligned}
h &= \frac{1000}{\pi r^2} = \frac{1000}{\pi \left(\sqrt[3]{\frac{500}{\pi}} \right)^2} \\
&= \frac{1000}{\pi} \left(\frac{\pi}{500} \right)^{\frac{2}{3}} = 2.\frac{500}{\pi} \left(\frac{\pi}{500} \right)^{\frac{2}{3}} \\
&= 2.500^{1-\frac{2}{3}}.\pi^{\frac{2}{3}-1} = 2.500^{\frac{1}{3}}.\pi^{\frac{-1}{3}} \\
&= 2\sqrt[3]{\frac{500}{\pi}} = 2r
\end{aligned}
$$

   Thus, S will be minimum when $r = \sqrt[3]{\frac{500}{\pi}}$ and $h = 2\sqrt[3]{\frac{500}{\pi}}$

9. Solve for y using integration by substitution: $y = \int 3t\sqrt{t^2+8}\, dt$.

   Solution:

   Let $t^2 + 8 = u \quad \Rightarrow \quad du = 2tdt \quad \Rightarrow \quad tdt = \frac{1}{2}du$

$$
\begin{aligned}
\int 3t\sqrt{t^2+8}\, dt &= 3\int \sqrt{u}\frac{1}{2}du \\
\int 3t\sqrt{t^2+8}\, dt &= \frac{3}{2}\int \sqrt{u}du \\
\int 3t\sqrt{t^2+8}\, dt &= \frac{3}{2}\left[\frac{u^{\frac{3}{2}}}{\frac{3}{2}} \right] + C \\
\int 3t\sqrt{t^2+8}\, dt &= (t^2+8)^{\frac{3}{2}} + C
\end{aligned}
$$

10. Evaluate the following integrals

$$(a) \int_1^2 \frac{t^3 + 3t^6}{t^4}\, dt \qquad (b) \int_0^1 (x^e + e^x)\, dx$$

Solution:

(a)

$$\begin{aligned} \int_1^2 \frac{t^3 + 3t^6}{t^4} dt &= \int_1^2 \left(\frac{1}{t} + 3t^2\right) dt \\ &= \left[\log(t) + t^3\right]_1^2 \\ &= (\log(2) + 8) - (\log(1) + 1) \\ &= 7 + \log(2) \end{aligned}$$

(b)

$$\begin{aligned} \int_0^1 (x^e + e^x) dx &= \left[\frac{x^{e+1}}{e+1} + e^x\right]_0^1 \\ &= \left[\frac{1^{e+1}}{e+1} + e^1\right] - [1] \\ &= \frac{1}{e+1} + e - 1 \\ &= \frac{1 + (e-1)(e+1)}{e+1} = \frac{1 + e^2 - 1}{e+1} \\ &= \frac{e^2}{e+1} \end{aligned}$$

11. Find the Riemann sum approximation for $\int_0^\pi \sin x\, dx$ using 4 intervals and left endpoints. Then do the same using 4 intervals and right endpoints. Find the actual value of this integral. How does the approximation compare to the actual value? You may use the approximate value 0.7 for $\sin(\pi/4)$ and $\sin(3\pi/4)$ and write your answer in terms of π.

Solution: Given $f(x) = \sin x$. $\Delta x = \frac{\pi - 0}{4} = \frac{\pi}{4}$.

$$\begin{aligned} L_n &= \sum_{i=0}^{n-1} f(x_i)\Delta x \\ R_n &= \sum_{i=1}^{n} f(x_i)\Delta x \end{aligned}$$

Here $x_i = x_0 + i\Delta x = \dfrac{\pi}{4} i$.
$x_0 = 0,\ \ x_1 = \frac{\pi}{4},\ \ x_2 = 2\frac{\pi}{4} = \frac{\pi}{2},\ \ x_3 = 3\frac{\pi}{4} = \frac{3\pi}{4},\ \ x_4 = \pi.$

$$L_n = \sum_{i=0}^{3} f\left(\frac{\pi}{4} i\right)\frac{\pi}{4} = \frac{\pi}{4}\sum_{i=0}^{3} f\left(\frac{\pi}{4} i\right)$$

$$
\begin{aligned}
&= \frac{\pi}{4}\left[f(0) + f\left(\frac{\pi}{4}\right) + f\left(\frac{\pi}{2}\right) + f\left(\frac{3\pi}{4}\right)\right] \\
&= \frac{\pi}{4}\left[\frac{1}{\sqrt{2}} + 1 + \frac{1}{\sqrt{2}}\right] = \frac{\pi}{4}\left[\frac{2+\sqrt{2}}{\sqrt{2}}\right] = \frac{\pi}{4}\left[\sqrt{2}+1\right] = \frac{\pi}{4}(2.4) = 0.6\pi \approx 1.88 \\
R_n &= \sum_{i=1}^{4} f\left(\frac{\pi}{4}i\right)\frac{\pi}{4} \\
&= \frac{\pi}{4}\sum_{i=1}^{4} f\left(\frac{\pi}{4}i\right) \\
&= \frac{\pi}{4}\left[f\left(\frac{\pi}{4}\right) + f\left(\frac{\pi}{2}\right) + f\left(\frac{3\pi}{4}\right) + f(\pi)\right] \\
&= \frac{\pi}{4}\left[\frac{1}{\sqrt{2}} + 1 + \frac{1}{\sqrt{2}}\right] = 0.6\pi \approx 1.88
\end{aligned}
$$

The actual integral value is :

$$
\int_0^{\pi} \sin x dx = -\cos x]_0^{\pi} = -\left[\cos pi - \cos 0\right] = -(-1-1) = 2
$$