9-9-2020 Notes, Calculus 1

Introduction to Limits

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Outline

Definition of e as a limit

Recall from 1.4:

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

To see why this is so, look at what happens as n keeps increasing in $A(1,n) = (1+(1/n))^n$:
NOTE: $n \to \infty$ just means we keep increasing n without bound.
$A(1,1) = 2$.

$n = 12 : A(1,12) = 2.613.$

$n = 365, A(1,365) = 2.714567482.$

NEVER GOES ABOVE $e = 2.71828......$

Another example

$$1 = \lim_{n \to \infty} 1 + \frac{1}{n}$$

To see why this is so, look at what happens as n keeps increasing in $f(n) = 1 + \frac{1}{n}$:

$f(1) = 2.$

$f(2) = 1 + (1/2) = 1.5.$

$f(10) = 1 + (1/10) = 1.1.$

$f(1000) = 1.001.$

NEVER GOES BELOW 1.

CAN GET AS CLOSE TO 1 AS WE WANT!

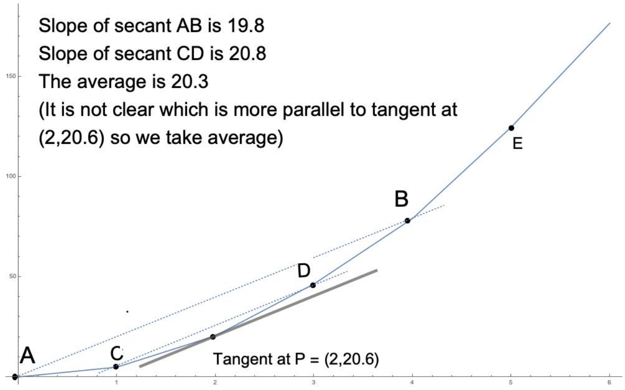
Finding slope using numerical approximation

Position of motorcyclist after accelerating from rest

t (secs)	0	1	2	3	4	5
s (feet)	0	4.9	20.6	46.5	79.2	124.8

So s is a function of t. s(0) = 0, s(1) = 4.9 and so on. We can approximate slope of tangent (instantaneous rate of change) at any value, say 2, by looking at the slope of the secants (average rates of change) passing through nearby points. Like (0,0) and (4, 79.2) or (1, 4.9) and (3,46.5). In this case we get those slopes as 19.8 and 20.8. See picture below.

Slope calculation



Where does the limit come in?

We estimated slope of tangent using slopes of secants.

What if we want exact value? Will need to use limits.

If, using equations, we can find slopes of ALL secant lines near the point,

then we can find slope of tangent line by looking at pattern:
WHAT VALUE ARE THEY APPROACHING?

Can we say there is a value THEY HAVE TO APPROACH?

Why can’t we use slope formula?

(Review) Why can’t we just use slope formula to find slope of tangent?

Crudely speaking, we can use the limit procedure to find values f(a) of a function f(x) when simply plugging in x=a does not work

Slope of parabola at $x = 1$

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = ?$$

This is really slope of secant from (1,1) to (x, x^2)
(On the graph of parabola $y = x^2$)
In other words, this is slope of secant from (1,1) to ANY other point (x, x^2) on the parabola.

Remember that you cannot just plug in x=1. (Why?)
Hint: How did we find slope of tangent line?

Slope of parabola at $x = 1$ - page 2

What we can do is to say

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} x + 1 = 2.$$

Why is it okay to cancel $x - 1$ from numerator and denominator?

So slopes of secants from $x = 1$ to ANY nearby point approach 2 !
So slope of tangent MUST BE 2.

Idea: look at what happens nearby!

When $x = 1.1, 1.01, 1.001$, etc.,$x + 1 = 2.1, 2.01, 2.001$, etc and we can see that $x + 1$ can be made as close to 2 as we want by making x close enough to 1.

KEY IDEAS

To look at a point

where you cannot plug in x value or at ∞

Look at what happens around it.

If limit exists, prove that

you can get as close to it as you want.

What do you mean “as close as possible”

What do you mean “We can make $x + 1$ as close to 2 as possible by choosing x close enough to 1?”

To make it precise, we say distance between $x + 1$ and 2 can be made as small as possible by choosing x close enough to 1. i.e, by making distance between x and 1 small enough.
To write “distance is as small as possible” symbolically we use the Greek letters ϵ and δ.
We say $|(x + 1) - 2| < \epsilon$ for any ϵ however small If we choose δ such that $|x - 1| < \delta$.
In practice this means solving for δ in terms of ϵ.

WARNING about just using numbers

Is it okay to find the limit by plugging in various values?
This would certainly help but it is not enough!
Example that shows why it is not enough:
$\sin(\pi/x) = 0$ for
$x = 1, x = 1/2 = 0.5, x = 1/3 = 0.333..., x = 1/4 = 0.25$,etc.,
But $sin(\pi/x)$ does not approach 0 as $x \to 0$.
In fact as we saw it keeps oscillating between 1 and -1

Warning: limit may not exist!

Note that the limit, here the slope of the tangent, may not always exist!

To find exact value, we look for a number L such that f(x) can be as close to L as possible.

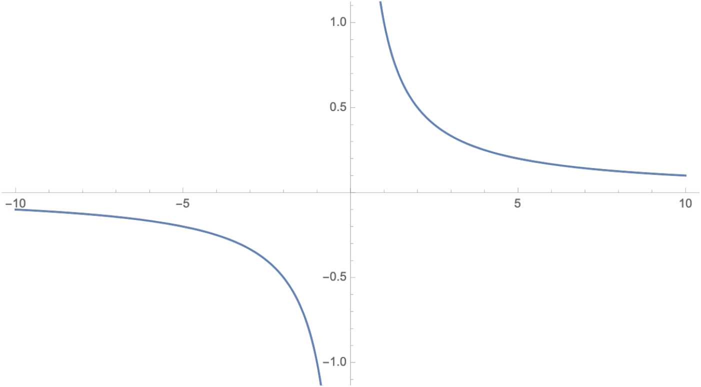
Situations where limit may not exist – singularity

1. f(x) goes to infinity
- graph has vertical asymptote

2. f(x) oscillates

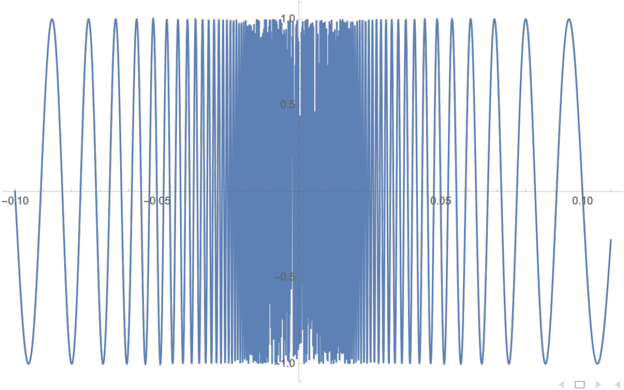
Example of singularity - asymptote

Graph of $y = 1/x$.
It has a vertical asymptote at $x = 0$. So $1/x \to \infty$ as $x \to 0$.



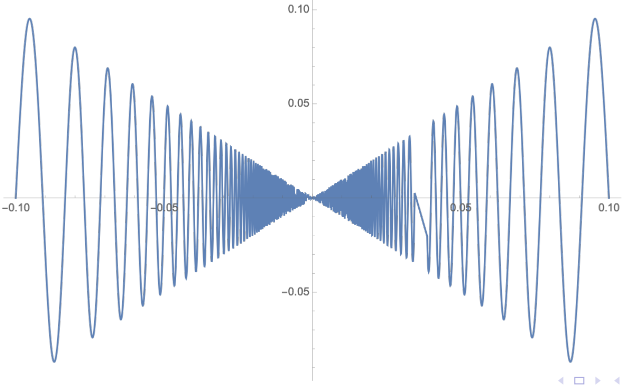
Example of singularity - oscillation

Graph of $y = \sin(\pi/x)$.
It oscillates from 1 to -1 infinitely many times near $x = 0$. So it has no limit as $x \to 0$.



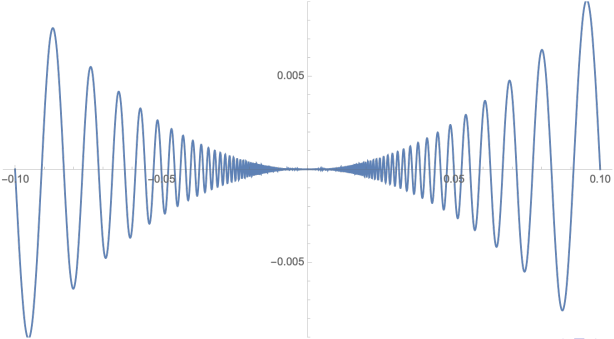
Example of limit in spite of oscillation

Graph of $y = x\sin(\pi/x)$.
It oscillates from 1 to -1 infinitely many times near $x = 0$. But it approaches 0 as $x \to 0$.



Why limit in spite of oscillation?

Graph of $y = x^2 \sin(\pi/x)$.
It oscillates from 1 to -1 infinitely many times near $x = 0$. But it approaches 0 as $x \to 0$.
Why? **Because it is squeezed between x^2 and $-x^2$**



Squeeze or Sandwich theorem

$$\text{If} \quad f(x) \leq h(x) \leq g(x)$$

$$f(x) \to L,\ g(x) \to L \quad \text{then} \quad h(x) \to L \quad \text{also.}$$

In plain English, if f(x) is squeezed between h(x) and g(x) and h and g approach the same value L then f also has to approach L because it has nowhere else to go.

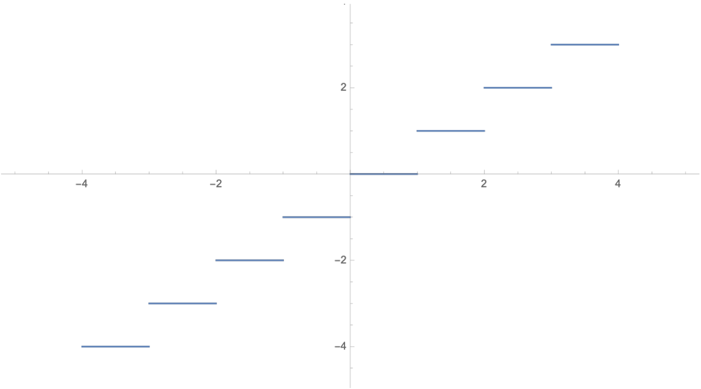
One sided limits

Sometimes a function may approach different values as x approaches a from the left and the right.

Example: already saw $1/x$ approaches ∞ from the right of 0 and $-\infty$ from the left of 0.

Here is another example: What is the value of the Floor function as x approaches 0 from the left?
(Recall: Floor(x) = Greatest integer $\leq x$)

Graph of Floor function



Limit of Floor function at 0

$$\lim_{x \to 0^+} Floor(x) = 0, \quad \lim_{x \to 0^-} Floor(x) = -1.$$

Read: Limit from right of 0 is 0, Limit from left of 0 is -1.

Note that,

if limit from either side does not exist
or if it approaches different limits from different sides,
limit does not exist.
So the limit of Floor function at 0
DOES NOT EXIST.

Practice problem: 2.2.13

Find the limit if it exists ; if it doesn't, explain why.

$$f(x) = \frac{1}{1+e^{1/x}}, \quad x \to 0.$$

Practice problem: 2.2.13 – Answer

$$f(x) = \frac{1}{1 + e^{1/x}}, \quad x \to 0.$$

As $x \to 0^+$, we see that $1/x \to \infty$.
So $e^{1/x} \to \infty$. Then $f(x) \to 0$.

As $x \to 0^-$, we see that $1/x \to -\infty$.
So $e^{1/x} \to 0$. Then $f(x) \to 1$.

Since it approaches different values from different sides, LIMIT DOESN'T EXIST.

Practice problem 2.3.21

Find the following limit, if it exists: $\lim\limits_{h\to 0} \dfrac{\sqrt{9+h}-3}{h}$

We simplify the expression by "rationalizing" the numerator.
We multiply above and below by "conjugate."

$$\frac{\sqrt{9+h}-3}{h} = \frac{\sqrt{9+h}-3}{h} \times \frac{\sqrt{9+h}+3}{\sqrt{9+h}+3}$$

$$= \frac{(\sqrt{9+h})^2-3^2}{h(\sqrt{9+h}+3)} = \frac{(9+h)-9}{h(\sqrt{9+h}+3)} = \frac{1}{\sqrt{9+h}+3}.$$

Therefore, $\lim\limits_{h\to 0} \dfrac{\sqrt{9+h}-3}{h} = \lim\limits_{h\to 0} \dfrac{1}{\sqrt{9+h}+3} = \dfrac{1}{6}.$