9-30-2020 Notes, Calculus 1

Product Rule for Derivatives

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Outline

Problem 3.1.56 , modified

$$Recall: \quad (e^x)' = e^x.$$

A particle is moving such that its position at time t is given by:

$$s(t) = 3e^t - 2t.$$

Find its velocity and acceleration at time t.
When is its direction (tangent to its path) horizontal?

Problem 3.1.56 , solution

Velocity is given by $s'(t)$:

$$v(t) = s'(t) = 3(e^t)' - 2(t^1)' = 3e^t - 2.$$

$$a(t) = v'(t) = s''(t) = (3e^t - 2)' = 3e^t.$$

When tangent is horizontal, its slope is 0. But slope of tangent is just derivative at that point. So we need $s'(t) = 3e^t - 2 = 0$.

$3e^t - 2 = 0 \implies 3e^t = 2 \implies e^t = 2/3 \implies t = ln(2/3) \simeq -0.4$

Since time is never negative, its direction is never horizontal.

LIST OF DIFFERENTIATION FORMULAE

$$(\textit{fixed number})' = 0$$

$$(\textit{power of } x)' = (x^r)' = rx^{r-1}$$

$$(\textit{natural exponential function})' = (e^x)' = e^x.$$

$$(\textit{constant times } f)' = (cf)' = c(f')$$

$$(\textit{Sum/difference of two functions})' = (f \pm g)' = f' \pm g'.$$

LIST OF DIFFERENTIATION FORMULAE – contd.

PRODUCT RULE: $(fg)' = fg' + f'g.$

QUOTIENT RULE: $(f/g)' = \dfrac{f'g - fg'}{g^2}.$

Example: xe^x

Let $f(x) = x$ and $g(x) = e^x$.
Then what is the derivative of $f(x)g(x) = xe^x$?

According to product rule, it is
$f'(x)g(x) + f(x)g'(x) = (x)'e^x + x(e^x)' = (1)e^x + xe^x = e^x + xe^x$.

Why is it so?

Why is $(xe^x)' = e^x + xe^x$ and NOT $(x)'(e^x)' = 1(e^x) = e^x$?
Let us look at the slope of the secants (average rate of change).
Remember that the limit of the slope of the secants (average rate of change) is the derivative.
Let $f(x) = xe^x$. Slope of secant is

$$= \frac{f(x+h) - f(x)}{h} = \frac{(x+h)e^{x+h} - xe^x}{h}$$

$$= \frac{xe^{x+h} - xe^x}{h} + \frac{he^{x+h}}{h} = \frac{xe^x e^h - xe^x}{h} + e^{x+h}$$

$$= (xe^x)\frac{e^h - 1}{h} + e^{x+h}$$

As $h \to 0$ this will equal $xe^x + e^x$. So derivative of xe^x will have two terms and not just one.

A quick idea about $f(x)g(x)$ in general

Let f, g be any two functions.

Key Task:

Write $\dfrac{f(x+h)g(x+h) - f(x)g(x)}{h}$ somehow in terms of $\dfrac{f(x+h) - f(x)}{h}$ and $\dfrac{g(x+h) - g(x)}{h}$.

A quick idea about $f(x)g(x)$ in general – page 2

The Main Idea:

Add and subtract $f(x)g(x+h)$.

You get $\dfrac{f(x+h)g(x+h)-f(x)g(x)}{h} =$

$$\frac{f(x+h)g(x+h)-f(x)g(x+h)+f(x)g(x+h)-f(x)g(x)}{h}.$$

Now pair up the first two and last two and take out common factors.

$$\frac{f(x+h)g(x+h)-f(x)g(x+h)+f(x)g(x+h)-f(x)g(x)}{h} =$$

$$g(x+h)\frac{f(x+h)-f(x)}{h}+f(x)\frac{g(x+h)-g(x)}{h}.$$

As $h \to 0$ you can see this will approach $g(x)f'(x)+f(x)g'(x)$.

Exercises on product rule

1. Differentiate $\sqrt{x}e^x$ using product rule.

[BTW how do you differentiate e^{x+3} etc., using what we know?]

2. What is $(fg)'(2)$, the derivative of $fg(x)$ at $x = 2$, if $f(2) = 1, g(2) = 3, f'(2) = 2, g'(2) = 1$?

Exercise on product rule - answer

$$(\sqrt{x}e^x)' = (\sqrt{x})'e^x + (\sqrt{x})(e^x)' = (x^{1/2})'e^x + \sqrt{x}e^x$$
$$= (1/2)x^{(1/2)-1}e^x + \sqrt{x}e^x = (1/(2\sqrt{x}))e^x + \sqrt{x}e^x.$$

BTW, To differentiate something like e^{x+3} proceed as follows:
Keep in mind that e^3 is just a number:

$$(e^{x+3})' = (e^x e^3)' = e^3(e^x)' = e^3(e^x) = e^{x+3}.$$

So the derivative of something like e^{x+3} is also itself.
But note that the same is not the case with e^{2x}, e^{x^2}, etc.,!!
We will need to use chain rule which we will study in 3.4.

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2. What is $(fg)'(2)$, the derivative of $fg(x)$ at $x = 2$, if
$f(2) = 1, g(2) = 3, f'(2) = 2, g'(2) = 1$?
We have $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$.
Plugging in $x = 2$, we get its derivative at 2:
$(fg)'(2) = f'(2)g(2) + f(x)g'(2) = 2 \times 3 + 1 \times 1 = 7$.

Problem 3.2.23

$$f(x) = \frac{x^2 e^x}{x^2 + e^x}. \;\; f'(x) = ?$$

Call $x^2 e^x = NUM$ and $x^2 + e^x = DEN$.
Then $f'(x) = [(NUM)' DEN - (DEN)' NUM]/(DEN^2)$.

Solution for 3.2.23

Now $NUM' = (x^2)'e^x + (x^2)(e^x)'$ *using product rule.* Simplifying, we get $NUM' = 2xe^x + x^2e^x = xe^x(2+x)$

$DEN' = (x^2 + e^x)' = (x^2)' + (e^x)' = 2x + e^x$.

Putting it all together, we get

$$f'(x) = \frac{(xe^x(x+2))(x^2+e^x) - (x^2e^x)(2x+e^x)}{(x^2+e^x)^2}$$

Taking the common term xe^x out we get

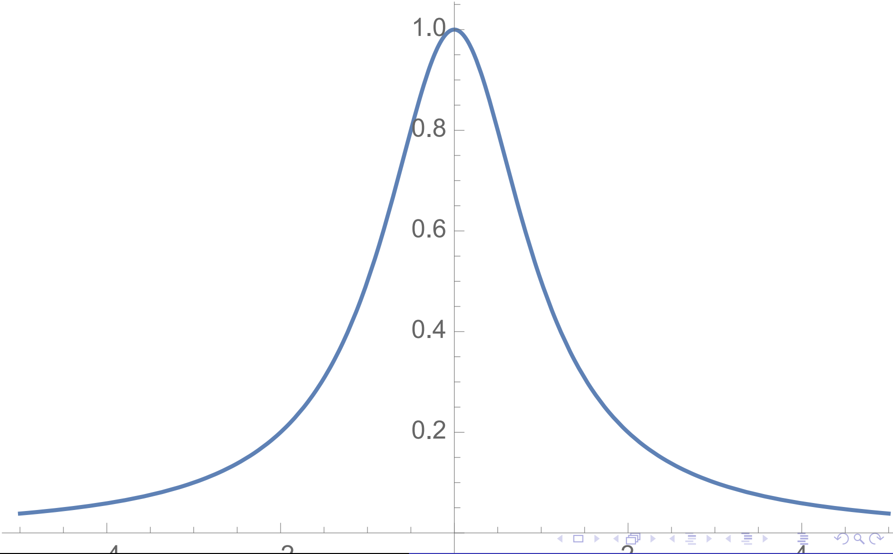
$$f'(x) = (xe^x)\frac{(x+2)(x^2+e^x) - x(2x+e^x)}{(x^2+e^x)^2}$$

$$= (xe^x)\frac{x^3 + xe^x + 2x^2 + 2e^x - 2x^2 - xe^x}{(x^2+e^x)^2} = \frac{e^x(x^3+2e^x)}{(x^2+e^x)^2}.$$

Problem 3.2.35: Witch of Maria Agnesi

The curve $y = \dfrac{1}{1+x^2}$ is called a Witch of Maria Agnesi. Find an equation of its tangent line at (-1,1/2).

Graph of the curve (Witch of Maria Agnesi)



Solution for 3.2.35

To differentiate use quotient rule:

$$dy/dx = \frac{(1)'(1+x^2) - 1(1+x^2)'}{(1+x^2)^2} = \frac{0-2x}{(1+x^2)^2} = -\frac{2x}{(1+x^2)^2}.$$

To get the slope at -1, we put -1 in the derivative:

$$f'(-1) = -2(-1)/(1+(-1)^2) = 2/4 = 1/2.$$

Using the point slope formula for the point (-1,1/2) we get the equation of the tangent at -1 as

$$(y-(1/2)) = (1/2)(x-(-1)) \implies y = (x/2)+1.$$