9-15-2020 Notes, Calculus 1

Continuity and Limits at infinity

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Outline

A rough definition of continuous

Crudely speaking, If the graph of the function has no breaks then the function is continuous.

(in other words, If you can draw it without taking the pencil out of the paper).

Mathematical definition of continuous

Earlier we said that, for functions such as $2x + 1$, the limit as $x \to a$ can be found by just plugging in $x = a$.

Here a is any random real number in the domain of $f(x) = 2x + 1$.

We said this is because function is continuous.

In fact, this is the very definition of continuous.

Precise definition of continuous

A function $f(x)$ is continuous AT $x = a$ if
The value that the function f(x) approaches as x approaches a must be the same as the value that you get it by plugging in to get $f(a)$.

$$\lim_{x \to a} f(x) = f(a).$$

If this is true at all values, we say that the function is continuous.

Note: First of all, both $f(a)$ AND $\lim_{x \to a} f(x)$ must be finite!

Examples of discontinuous functions

Can you think of some examples of discontinuous functions?

Examples:

1. $f(x) = \dfrac{1}{1-x}; \ at\ x = 1.$
2. The Step function $Floor(x)$.
3. Any piecewise function with a break, such as
$$f(x) = \begin{cases} x^2, \ x \leq 0 \\ x+1, \ x > 0 \end{cases}$$

Visualizing continuity

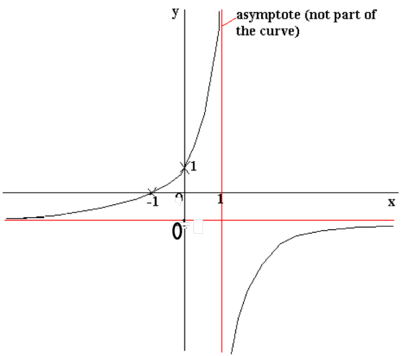
What is really going on when we “graph” a function?
We plot points, join them into a curve, and *just assume* that at the other points the function will have values indicated by the curve !

For a discontinuous function like the step function, you can't just join values!

Ways in which $f(x)$ can be discontinuous at a point

1. The function is not defined at $x = a$. That is, $f(a)$ is undefined or the function goes to infinity.
2. $f(a)$ is well defined but the Limit does not exist as $x \to a$.
3. $$\lim_{x \to a} f(x) \neq f(a).$$

Example 1: Graph of $1/(1-x)$



Why is it discontinuous?

Why is $1/(1-x)$ discontinuous?

Where is it discontinuous?

Answer: It is discontinuous at $x = 1$.
Here denominator is zero and it has a vertical asymptote.
Limit from left is $+\infty$ and from right limit is $-\infty$.

Example 2: From Fall 2019 Quiz 4

Let

$$f(x) = \begin{cases} x - 1, & x \leq 0, \\ x + 1, & x > 0. \end{cases}$$

a)(6 points) Does $\lim_{x \to 0} f(x)$ exist? If it does, say what it is.

b) (4 points) Is the function continuous at 0? Must use definition of continuity involving limits.

Example 2: From Fall 2019 Quiz 4 -Solution

a) First we find $\lim_{x \to 0^+} f(x)$ by looking at what happens to the right of 0. Here the function is defined by $x - 1$ which approaches -1 as x approaches 0. It is a polynomial, so we can find limit by just plugging in $x = 0$. So the limit is -1.

To find $\lim_{x \to 0^-} f(x)$ we look at what happens to the left of 0.

Here the function is defined by $x + 1$ which is approaches 1 as x approaches 0. The function $x + 1$ is a polynomial also, so we can find its limit also by just plugging in.
So the limit is 1.

Since the two limits are not equal, $\lim_{x \to 0} f(x)$ does not exist.

Example 2: From Fall 2019 Quiz 4 -Solution -continued

b) We have $f(0) = 0 - 1 = -1$ because $0 \ < \ -1$ and we need to use $f(x) = x - 1$.

But this is not equal to $\lim_{x \to 0} f(x)$ since the limit doesn't even exist; so it is NOT continuous at 0.

The graph will have a jump from -1 to 1 at $x = 0$. It will be a broken straight line.

Removable discontinuity

Recall that $(x^2 - 1)/(x - 1)$approaches 2 as $x \to 1$.

So even though $f(1)$ cannot be found by plugging in $x = 1$, if you define $f(1) = 2$, then the function is continuous at $x = 1$.

A point where the function can be made continuous by defining it properly is called a **removable discontinuity.**

How would you define $x \sin(\pi/x)$ at $x = 0$ to make it continuous?

Intermediate Value Theorem – Statement

Intermediate value theorem

If a function is continuous from $x = a$ to $x = b$, then it takes every value between $f(a)$ and $f(b)$ for some x between a and b.

Graphically, between the horizontal lines $x = a$ and $x = b$, EVERY horizontal line will intersect the graph.

Intermediate Value Theorem – Application 1

How do we know that there is a real number x that is the square root of 2? Note: We do know it cannot be a fraction a/b.

One way is to show geometrically. If you make a square with corners at (0,0), (0,1), (1,0) and (1,1) then its diagonal can be shown to be between 1 and 2. Of course we know it is the square root of 2 because $2 = 1^2 + 1^2 = c^2$.

Another way is to say that it is a root of $x^2 - 2$ and this polynomial (actually quadratic) has value -1 at 1 and 2 at 2. So it must be 0 somewhere between 1 and 2 because 0 is between -1 and 2.

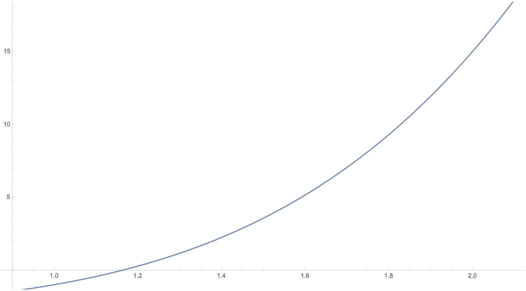
Intermediate Value Theorem – Application 2

Example: Problem 2.5.53:
Show that $x^4 + x - 3 = 0$ has a root in $(1, 2)$ (for some x in (1,2) this equation is satisfied)

Answer: $f(1) = -1$ is negative, $f(2) = 15$ is positive. So at some x in (1,2) function must cross $x-$axis. In other words, since 0 is between -1 and 15, by the theorem some x in (1,2) must have $f(x) = 0$.

Graphical view of IVT

You can see that graph crosses x-axis between 1.15 and 1.2. You could have figured it out by showing $f(1.15) < 0$ and $f(1.2) > 0$. Continuing like this, you can get a good approximation for the root (aka zero aka solution of $x^4 + x - 3 = 0$).



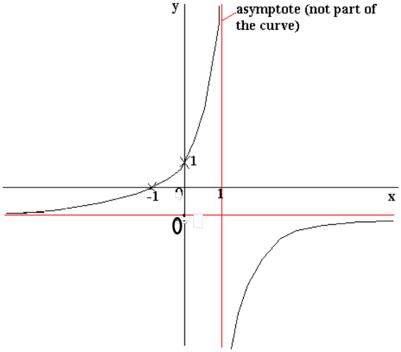
2.6 Limits at infinity

Finding limit at ∞ (or $-\infty$) is basically looking at what happens as $|x|$ gets larger and larger.

You can plug in large values or, if you have no calculator, think about what happens as $|x|$ gets larger and larger.

Example: old friend $1/(1-x)$

We know this graph has vertical asymptote at $x = 0$. Where does it have Horizontal asymptote?



Horizontal asymptote of $1/(1-x)$

As x goes to ∞ (or $-\infty$), the denominator of $1/(1-x)$ is getting larger and larger, therefore the fraction itself gets smaller and smaller, getting as close to 0 as we want. So we say the limit is 0, although it never equals zero.

Remember, a fraction can only be zero if numerator is zero.

So the horizontal asymptote is the $x-$axis, in both directions.

As a general rule, e^{kx} with positive k and all polynomials with a positive leading term (i.e, starting as $ax^n + bx^{n-1} + ...$) will go to infinity as $x \to \infty$. (If a is negative it will go to $-\infty$).

Limit of $\sin x$ at infinity

What happens to $\sin x$ as $x \to$ *infty*?

Answer: $\sin x$ is an oscillating function. It has no limit as $x \to \infty$ or $-\infty$.

NOTE: $\sin(1/x)$ will go to what as $x \to \infty$ or $-\infty$?

Limit of functions with numerator and denominator

In general for functions with numerator and denominator their behavior is dictated by the fastest growing part.

If it involves polynomial and exponential functions, the exponential functions might dictate the behavior.

For rational functions, the highest power of x will tell us how it behaves.

Usually dividing above and below by highest power or the one with the highest positive exponent will work.

Rational functions (polynomials in numerator and denominator)

Example (2.6.15)

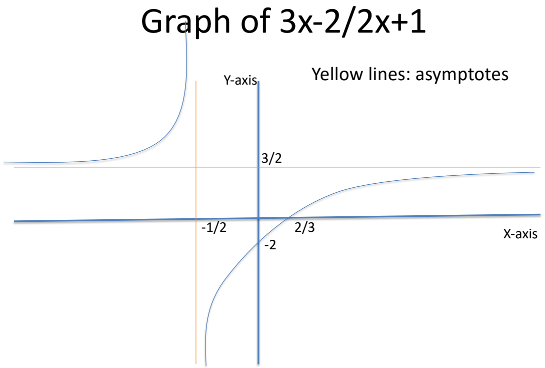
$$\lim_{x \to \infty} \frac{3x - 2}{2x + 1}$$

The coefficient of x tells us it approaches 3/2 because when x is very big the other terms contribute much less.

Here we can divide by x everywhere to get $(3 - \dfrac{2}{x})/(2 + \dfrac{1}{x})$.

Both $2/x$ and $1/x$ go to zero as $x \to \infty$.

Graph of $(3x-2)/(2x+1)$



Shortcut when dealing with polynomials

If highest power is in numerator, function will go to infinity as x goes to infinity (plus or minus depending on coefficient). It will go to zero if denominator has highest power.

If highest powers are same, we can just divide by the coefficients.
For instance, in $(3x - 2)/2x + 1)$ we could have said limit is 3/2 by looking at coefficients.

WARNING: **This will only work for polynomials**, and when x goes to infinity. Also limit depends on coefficients. For these reasons, better not to use shortcuts.

Practice problem : Fall 2019 quiz 4

Find the limits:

$$\lim_{x \to \infty} \frac{2x^2 + 1}{x^2 + x - 1}$$

What are the vertical and horizontal asymptotes?

Solution to practice problem : Fall 2019 quiz 4

Solution: The highest power above and below is x^2. So you can use the shortcut and just find limit of $2x^2/x^2 = 2$.
But as mentioned earlier, I recommend using longer method.

$$\lim_{x\to\infty} \frac{2x^2+1}{x^2+x-1} = \lim_{x\to\infty} \frac{(2x^2+1)/x^2}{(x^2+x-1)/x^2}$$

$$= \lim_{x\to\infty} \frac{2+\frac{1}{x^2}}{1+\frac{1}{x}-\frac{1}{x^2}} = \frac{2}{1} = 2.$$

For vertical asymptotes see where denominator is zero.
Answer: $x = (-1 \pm \sqrt{5})/2$.
For horizontal asymptotes look at limit at infinity. $y = 2$ is the horizontal asymptote

Practice problem: 2.6.24

$$\lim_{x\to\infty} \frac{\sqrt{1+4x^6}}{2-x^3}$$

The actual highest power of x in the numerator is x^3. When you divide the numerator by x^3 and take it inside the square root it will become x^6.

Note: You can also do this by ignoring all except highest powers: you will get $\sqrt{4x^6}/-x^3$ which simplifies to -2. But the longer way is preferable and more precise mathematically. The shortcut works only when higher power is same above and below.

Practice problem: 2.6.24 – solution

$$\lim_{x\to\infty} \frac{\sqrt{1+4x^6}}{2-x^3} = \lim_{x\to\infty} \frac{\frac{\sqrt{1+4x^6}}{x^3}}{\frac{2-x^3}{x^3}} = \lim_{x\to\infty} \frac{\sqrt{\frac{1+4x^6}{x^6}}}{\frac{2}{x^3}-1}$$

$$= \lim_{x\to\infty} \frac{\sqrt{\frac{1}{x^6}+4}}{\frac{2}{x^3}-1} = \frac{\sqrt{4}}{-1} = -2.$$

In the last line we use the fact that $1/x^6$ and $2/x^3$ both go to zero as $x \to 0$.