8-26-2020 Notes, Calculus 1

Introduction to Calculus 1

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Outline

INTRODUCTION TO DERIVATIVES

Derivative: Finds **instantaneous rate of change** of a function that is **changing continuously.**

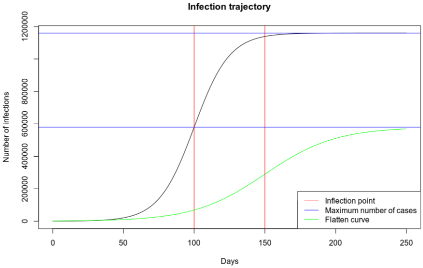
Special Case (Application) : Derivative is the slope of the tangent of a curve at a point.

You can say it tells you the direction in which the curve is going at a given point.

The slope of the curve can tell you a lot about how the function is behaving.

CORONAVIRUS CASES TRAJECTORY

(image thanks to stories.marquette.edu)
Example of EXPONENTIAL GROWTH (at least in beginning)
Can you tell just from the slopes at different points when a) Cases are starting to flatten out? b) Where the infections start to stop "climbing" ?
What are the dependent and independent variables?



Functions and slopes

You can also tell what kind of function it is from the slope.

What kind of function has a constant rate of growth?
i.e, increase in y /increase in x is the same at all points?

What kind of function has growth rate proportional to value of x ?

What kind of function has growth rate proportional to value of function itself (i.e, y value ?)

A crude calculation of derivative

The change in value of y when x changes by 1 gives a crude idea of derivative / slope of curve at a given value of x. In fact, this is used often in business and economics and is called marginal growth.

Linear function:

$$y = f(x) = mx + b \ : \ f(x+1) - f(x) = m.$$

Quadratic function:

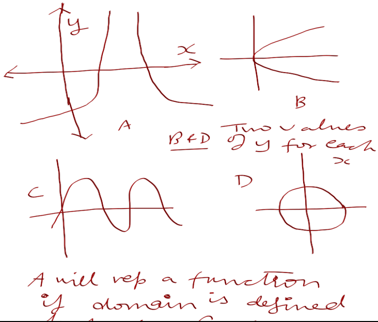
$$y = f(x) = kx^2 \ : \ f(x+1) - f(x) = 2kx + k \simeq 2kx \text{ (when } x \text{ is big)}$$

Exponential function:

$$y = f(x) = 2^x \ : \ f(x+1) - f(x) = 2^{x+1} - 2^x = 2^x = y.$$

Definition of functions and one-one-ness

Function: Each input value results in only one output value.
One-one function: Each output value comes from only one input value.
Function has an inverse if and only if it is one-one.
Which of the following are functions? Which are one - one?



Piecewise defined function

Problem 41 from 1.1

Question: Find $f(-3), f(0), f(1)$ for the following function and graph it:

$$f(x) = \begin{cases} x+1 & x \leq -1 \\ x^2 & x > -1 \end{cases}$$

For $f(-3)$ we need to use $y = x + 1$ because $-3 \leq -1$.
We get $f(-3) = -3 + 1 = -2$.

For $f(0)$ we need to use $y = x^2$ because $0 > -1$.
We get $f(0) = 0^2 = 0$.

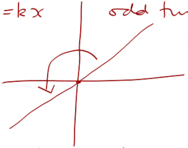
For $f(1)$ also we need to use $y = x^2$ because $1 > -1$.
We get $f(1) = 1^2 = 1$.

Even and odd functions

Symmetry about $y-$axis and (0,0).
Even functions are symmetric about $y-$axis.
The expression for y is unchanged when x is replaced by $-x$.
Odd functions are symmetric about (0,0).
The expression for y becomes negative when x is replaced by $-x$.

Examples:

$y = kx$ odd function



$x \leftrightarrow -x$

$y = k(-x)$

$= -kx$

$= -y$

What about $y = x+1$

$y(-x) = -x+1 \neq -(x+1)$

Not odd — not symmetric about (0,0)

Even function: Symmetric about y axis

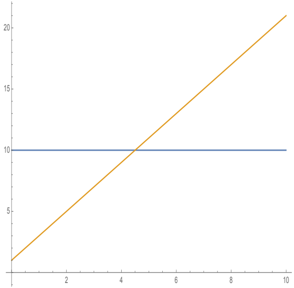
$y = x^2$ or x^4 or $x^6 + x^2$

all even

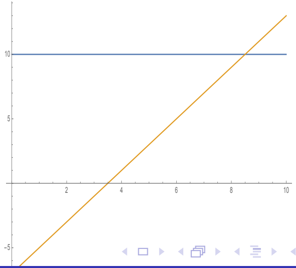
when $x \to -x$ function is unchanged

Translation

EQUATION OF LINE
$y = 2x + 1$

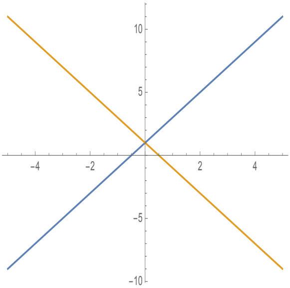


EQUATION OF TRANSFORMED LINE
$y = 2(x - 4) + 1$.
x is changed to $x - 4$. [So $2(x - 4) + 1 = 2x - 7$].
Graph of $y = 2x + 1$ is shifted to the right by 4.



Reflection

If x is replaced by $-x$, reflect graph about $y-$axis
If y is replaced by $-y$, reflect graph about $x-$axis
EQUATION OF LINE $y = 2x + 1$ and $y = -2x + 1$
x is replaced by $-x$
Blue: $2x + 1$ Yellow: $-2x + 1$



Rules for transformations

GUIDELINES FOR TRANSFORMING GRAPHS
RULE:

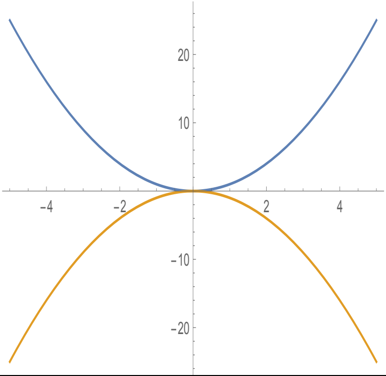
If x is replaced by $x+k$, shift graph to left
If x is replaced by $x-k$, shift graph to right

If x is replaced by $-x$, reflect graph about $y-$axis
If y is replaced by $-y$, reflect graph about $x-$axis

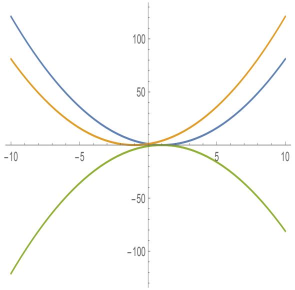
NOTE: When x or y are *multiplied* by a number, graph is *shrunk* or *expanded*

EXAMPLE: PARABOLA

Blue line: $y = x^2$; Yellow line: $y = -x^2$
Parabola is reflected about $x-$axis as $y(= x^2)$ is replaced by $-y(= -x^2)$



Translation followed by reflection



Graph of $y = (x-1)^2$(blue) obtained by moving $y = x^2$ (above) to the right by 1. (translation)

Graph of $y = (-x-1)^2 = (x+1)^2$ (yellow) obtained by reflecting $y = (x-1)^2$ about the $y-$axis because x is replaced by $-x$.

What is another way of getting the yellow graph?

Graph of $y = -(x-1)^2$ (green) obtained by reflecting $y = (x-1)^2$ about the $x-$axis because $y = (x-1)^2$ is replaced by $y = -(x-1)^2$.

COMBINING DIFFERENT OF FUNCTIONS

HOW DO YOU COMBINE FUNCTIONS

REMEMBER: A FUNCTION IS A RULE OR PROCEDURE

IF YOU KNOW VALUE OF ONE THING (SAY x)
IT TELLS YOU FIND VALUE OF A RELATED THING (SAY y).

Example: $y = C(x)$ is the cost of producing x units.
y represents cost and x represents number of units.

WHEN YOU COMBINE FUNCTIONS TO GET A NEW ONE,
YOU ARE REALLY SAYING HOW THE NEW RULE WORKS.

Example

Let $f(x) = x^2$. Let $g(x) = 2x + 1$.
Then $f(x) + g(x) = x^2 + 2x + 1$ is a new function called $f + g$

f says “square the value”; g says “multiply by 2 then add 1”

$f + g$ says: “For every x add $f(x)$ and $g(x)$.

This results in “square x then add 2 times x then add 1.

Rules for combining functions

FIVE WAYS OF COMBINING FUNCTIONS

1. *SUM* : $(f+g)(x) = f(x) + g(x)$
   "ADD VALUES AT EACH x"
2. *DIFFERENCE* : $(f-g)(x) = f(x) - g(x)$
   "SUBTRACT VALUES AT EACH x"
3. *PRODUCT* : $fg(x) = f(x) \times g(x)$
   "MULTIPLY VALUES AT EACH x"
4. *QUOTIENT* : $f/g(x) = f(x)/g(x)$
   "DIVIDE VALUES AT EACH x, AS LONG AS $g(x) \neq 0$."
5. *COMPOSITION* : $f \circ g(x) = f(g(x))$.
   "PLUG IN VALUE OF $g(x)$ INSTEAD OF x IN $f(x)$

Example

APPLYING THESE RULES TO GIVEN f and g

$(f+g)(x) = f(x) + g(x) = x^2 + 2x + 1$
$(f-g)(x) = f(x) - g(x) = x^2 - (2x+1) = x^2 - 2x - 1.$
$fg(x) = f(x) \times g(x) = x^2(2x+1) = 2x^3 + x^2$
$f/g(x) = f(x)/g(x) = \dfrac{x^2}{2x+1}$
$f \circ g(x) = f(g(x))$ "PLUG IN VALUE OF $g(x)$ INSTEAD OF x IN $f(x)$ AT EACH x"
$f(g(x)) = f(2x+1) = (2x+1)^2 = 4x^2 + 4x + 1.$

Another example

Find the value of each of above combinations at -1,0,1, 2.
Do it two ways:
1. Find $f+g$, fg, etc. directly
by plugging into expressions that we found above for them.
2. Find $f(x), g(x)$ separately, then combine values.

We are doing this just as a way to reinforce the idea that the combination of a function is really just a rule that you apply to each value of x.
You only need to use one of these ways to do it. If you don't need to find $f/g(x)$ etc., and just need $f/g(-1)$ then way 1 is easier.

Example with numbers

Let $x = -1$. Let us find $f/g(-1)$ first in the two ways. *Way* 1 :
Put $x = -1$ *in* $\frac{x^2}{2x+1}$: $f/g(-1) = \frac{(-1)^2}{2(-1)+1} = \frac{1}{-1} = -1.$
Way 2 : $f(-1) = (-1)^2 = 1; g(-1) = 2(-1) + 1 = -1$. $f/g(-1) = 1/(-1) = -1.$

The difference is more apparent when you try to find $f \circ g(-1)$.
Way 1 : *Put* $x = -1$ *in* $f(g(x)) = 4x^2 + 4x + 1.$: $f/g(-1) = 4(-1)^2 + 4(-1) + 1 = 4 - 4 + 1 = 1$. *Way* 2 : $f(x) = x^2; g(-1) = -1; f(g(-1)) = (g(-1))^2 = (-1)^2 = 1.$
If you want to find $f \circ g(0)$ then
$f \circ g(0) = f(g(0)) = f(1) = 1^2 = 1.$
So basically for each value of x you can plug in x and find value of $f \circ g(x)$.

PRACTICE PROBLEM

DO ALL OF ABOVE FOR $f(x) = x + 1, g(x) = 1/(3x - 2)$
ALSO FIND $g \circ f(x), g \circ g(x)$
What values of x are not acceptable for these combinations?
(i.e, what values of x are not in domain).

ANSWERS

$$f + g(x) = f(x) + g(x) = (x + 1) + \frac{1}{3x - 2}.$$

This can be written as one fraction, if necessary.
The value $x = 2/3$ is not valid for this because it makes denominator zero.

Practice problem – page 1

$$f - g(x) = f(x) - g(x) = (x+1) - \frac{1}{3x-2}.$$

This can be written as one fraction, if necessary.
The value $x = 2/3$ is not valid for this because it makes denominator zero.

$$fg(x) = f(x) \times g(x) = (x+1)\frac{1}{3x-2} = \frac{x+1}{3x-2}$$

The value $x = 2/3$ is not valid for this because it makes denominator zero.

Practice problem – page 2

$$f/g(x) = f(x)/g(x) = (x+1)/\left(\frac{1}{3x-2}\right)$$

$$= (x+1)\left(\frac{3x-2}{1}\right) = 3x^2 + x - 2.$$

The value $x = 2/3$ is not valid for this because it makes denominator of g zero.
This is true *even though* the final expression has no denominator.
In order to start evaluating f/g we need $g(x)$ to be defined in first place.

Practice problem – page 3

$f \circ g(x) = f(g(x))$ PLUG IN VALUE OF $g(x)$ INSTEAD OF x IN $f(x)$

$f(g(x)) = f(\frac{1}{3x-2}) = (\frac{1}{3x-2} + 1)$.

The value $x = 2/3$ is not valid for this because it makes denominator of g zero.

$g \circ f(x) = g(f(x))$ PLUG IN VALUE OF $f(x)$ INSTEAD OF x IN $g(x)$

$g(f(x)) = g(x+1) = \frac{1}{3(x+1)-2} = \frac{1}{3x+1}$.

You can see that $x = -1/3$ will make the final expression undefined.

So domain is all real numbers except $-1/3, 2/3$.

Practice problem – page 4

$g \circ g(x) = g(g(x))$ PLUG IN VALUE OF $g(x)$ INSTEAD OF x IN $g(x)$

$$g(g(x)) = g(1/(3x-2)) = \frac{1}{3(1/(3x-2))-2}$$

$$= \frac{1}{\frac{3}{3x-2}-2} = \frac{1}{\frac{3-2(3x-2)}{3x-2}} = \frac{1}{\frac{7-6x}{3x-2}} = \frac{3x-2}{7-6x}.$$

The value $x = 2/3$ is not valid for this because it makes denominator of g zero.

This is true *even though* the final expression will work fine with $x = 2/3$.

In order to start evaluating $f \circ g$ we need $g(x)$ to be defined in first place.

Application of combination

A bicycle wheel turns at a rate of 80 revolutions per minute (rpm).
Write a function that represents the number of revolutions $r(t)$ in t minutes.
For each revolution of the wheels, the bicycle travels 7.2 ft.
Write a function that represents the distance traveled $d(r)$ (in ft) for r revolutions of the wheel.
Find $d \circ r(t)$ and interpret the meaning in the context of this problem.
Evaluate $d \circ r(30)$ and interpret the meaning in the context of this problem.

Answer for application problem

This is an application of composition of functions.
In order to find the find the distance travelled per minute, we need to first find revolutions per minute and then multiply each revolution by circumference of wheel, which is same as distance travelled in one revolution.
So $d(r) = 7.2r$. This is the distance travelled after r revolutions.
Now $r(t) = 80t$ because there are 80 revolutions per minute and after t minutes it does $80t$ revolutions.
So $d(t) = d \circ r(t) = d(r(t)) = 7.2(r(t)) = 7.2(80t) = 576t$.
This is a rather simple example but there are situations where compositions are really useful in breaking down complex functions.