11-20-2020 Notes, Calculus 1

(5.5) Integration using Subsitution

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Outline

Antiderivative of e^{2x}, $\cos x$ and $\sin 2x$

To figure out what is the antiderivative of e^{2x} let us see how we differentiate e^{2x}.
We wrote $e^{2x} = e^u$ with the **substitution** $u = 2x$.

$$\frac{d(e^u)}{dx} = \frac{d(e^u)}{du} \times \frac{du}{dx} = e^u \times 2 = 2e^{2x}.$$

To find the anti-derivative, we again use the same substitution.
Now $u = 2x \implies du = (du/dx) \times dx = 2dx \implies dx = du/2$.

$$\int e^u \, dx = \int e^u \frac{du}{2} = \frac{1}{2} \int e^u \, du = \frac{e^u}{2} + C = \frac{1}{2} e^{2x} + C.$$

What are the antiderivatives of $\cos(kx)$ and $\sin(kx)$?

Substitution in a nutshell

Answer to question:
$\int \cos(2x)\, dx = \sin(2x)/2 + C, \ \int \sin(2x)\, dx = -\cos(2x) + C.$

Integration by Substitution is just doing the **chain rule backwards.**

If it looks like the function could be obtained by differentiation using chain rule, then substitution might be a good idea.

IMPORTANT: Convert everything to a function of the substitution variable u

Antiderivative of $\tan x$

Recall how we got $\tan x = \sin x / \cos x$
in the derivative of $\ln \cos x$.
Using chain rule (with $u = \cos x$) we got the derivative as
$(1/\cos x)(\cos x)' = -\sin x/\cos x = -\tan x$.
So we get that the antiderivative of
$\tan x = \int \tan x \, dx = -\ln \cos x + C = (-1) \ln \cos x + C =$
$\ln(\cos x)^{-1} + C = \ln(1/\cos x) + C = \ln(\sec x) + C$.

Can you guess the antiderivative of xe^{x^2} ?

Antiderivative of xe^{x^2}

We see that x is $1/2$ times derivative of x^2.
When you use chain rule, you would get $e^{x^2}(2x)$ when you differentiate e^{x^2}.
Setting $u = x^2$ we will get derivative of $e^u = e^u(u') = 2xe^{x^2}$.
So antiderivative of xe^{x^2} is $(1/2)e^{x^2}$.

Key idea: Recognize u and du

Key step: Write ALL in terms of u

Here $u = x^2$. Then $2x\ dx = du \implies x\ dx = du/2$.

$$\int xe^{x^2}\ dx = \int e^u \frac{du}{2} = \frac{1}{2}\int e^u\ du + C = \frac{e^u}{2} + C = e^{x^2}/2 + C.$$

Antiderivative of $x^2 \; sinx^3$

$$\int_0^1 (x^2)\sin(x^3)dx$$

Key Point: Change limits to the values of u also
[*Or use the limits AFTER integrating and putting x back in*]

Let $x^3 = u$. Then $du = 3x^2\, dx$ and $x^2\, dx = du/3$.
We need to change the limits from x to u. Also
$u(0) = 0^3 = 0, u(1) = 1^3 = 1$.

$$So \int_0^1 x^2 \sin(x^3)\; dx = \int_0^1 \sin u\; du/3 = (1/3)\left[-\cos u\right]_0^1$$

$$= (-\cos(1) - (-1))/3 = (1 - \cos 1)/3.$$

PROBLEM 5.5.7

$$\int x\sqrt{1-x^2}\,dx$$

Let $1-x^2 = g(x) = u$. Then $du = -2x\,dx$ and $x\,dx = du/(-2)$.

$$\int x\sqrt{1-x^2}\,dx = \int \sqrt{u}\frac{du}{-2} = \int u^{1/2}\,du = -\frac{1}{2}\left[\frac{u^{3/2}}{3/2}\right] + C$$

$$= -\frac{1}{2}\left[\frac{2}{3}(1-x^2)^{3/2}\right] + C = -\frac{1}{3}(1-x^2)^{3/2} + C.$$

Problem 5.5.38

$$\int \frac{1}{\cos^2 t\sqrt{1+\tan t}}\,dt$$

Key is to write $1/\cos^2 t$ as $sec^2 t$ so that we recognize both $\tan t$ and its derivative $sec^2 t$ as present in same expression.

$$u = 1+\tan t \implies du = sec^2 t\;dt$$

AND

$$\int \frac{1}{\cos^2 t\sqrt{1+\tan t}}\,dt = \int \frac{sec^2 t}{\sqrt{1+\tan t}}\,dt$$

$$= \int \frac{du}{\sqrt{u}} = \int u^{-1/2}\,du = \frac{u^{1/2}}{1/2} + C = 2\sqrt{1+\tan t} + C.$$

PROBLEM 5.5.30

$$\int \frac{\sec^2 x}{\tan^2 x}\, dx$$

Let $u = \tan x$. Then $du = \sec^2 x\, dx$ and

$$\int \frac{\sec^2 x}{\tan^2 x}\, dx = \int du/u^2 = (-u^{-1}) + C = (-1/\tan x) + C.$$

PROBLEM 5.5.46

$$\int x^3\sqrt{x^2+1}\,dx$$

Let $u = x^2 + 1$. Then $du = 2xdx$ and $xdx = du/2$. So $x^3dx = x^2(xdx) = (u-1)du/2$ and

$$\int x^3\sqrt{x^2+1}\,dx = \int \sqrt{u}(u-1)(du/2) = \frac{1}{2}\int (u^{3/2} - u^{1/2})\,du =$$

$$\frac{1}{2}\left(\frac{u^{5/2}}{5/2} - \frac{u^{3/2}}{3/2}\right) + C = \frac{(x^2+1)^{5/2}}{5} - \frac{(x^2+1)^{3/2}}{3} + C.$$

PROBLEM 5.5.63

$$\int_0^{13} \frac{dx}{\sqrt[3]{(1+2x)^2}}$$

Let $u = 1 + 2x$. Then $dx = du/2$. $u(0) = 1$, $u(13) = 27$. We get

$$\int_0^{13} \frac{dx}{\sqrt[3]{(1+2x)^2}} = \int_1^{27} \frac{du/2}{u^{2/3}}$$

$$= \frac{1}{2}\left[\frac{u^{1/3}}{1/3}\right]_1^{27} = (3/2)(27^{1/3} - 1^{1/3}) = (3/2)(3-1) = 3.$$