11-2-2020 Notes, Calculus 1

Graphing curves using derivatives

Sankar Sitaraman

Math Dept, Howard University

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Outline

Review of Graphs and Derivatives

You can tell a lot about the graph of a function by looking at the derivatives.

At points where function is increasing or decreasing derivative is positive or negative. Where derivative is increasing or decreasing second derivative is positive or negative.

You can also say something about how many $x-$intercepts graph will have. Conversely, you can also say something about the derivative by looking at the values of the function.

Review of some facts about derivative

1. If function is increasing then derivative is positive.
2. If derivative is positive slope of tangent is positive (going up).
3. If function is decreasing then derivative is negative.
4. If derivative is negative slope of tangent is negative (going down).
5. At peak or valley tangent is horizontal (derivative is zero).

Review of some facts about derivative – page 2

1. At peak, function goes from increasing to decreasing.
2. At peak, derivative goes from positive to negative.
3. At valley, function goes from decreasing to increasing.
4. At valley, derivative goes from negative to positive.
5. At inflection point, derivative is zero but function is not maximum or minimum. Instead it keeps increasing or keeps decreasing.

Second derivative and Concavity

Geometrically speaking, second derivative tells us **how the slope is changing.**

$f''(x)$ is the derivative of $f'(x)$ which is the slope of the tangent.

Since derivative is rate of change, $f''(x)$ is rate of change of slope. **How slope is changing tells us how the graph is curved.**

Second derivative and Concavity – page 2

1. If slope is increasing then graph is concave up (holds water) and $f''(x)$ is positive (because slope is increasing).
2. If slope is decreasing then curve is concave down (can't hold water) and $f''(x)$ is negative (because slope is decreasing)
3. At peak, graph is concave down. So $f''(x)$ is negative.
4. At valley, graph is concave up. So $f''(x)$ is positive.
5. At inflexion point, $f''(x) = 0$ AND concavity changes.
6. Concavity changes at inflexion point-means $f''(x)$ changes sign as you pass that point.

Example problem from test 2: $\sin x - x$

PROBLEM : Graph $f(x) = \sin x - x$.
Find where it is increasing, decreasing, local extrema, absolute extrema, concavity, inflexion points and asymptotes.

Where are $f(x), f'(x), f''(x)$ equal to zero?
$f(x) = \sin x - x \implies f'(x) = \cos x - 1$.
$f'(x) = 0 \implies \cos x = 1 \implies x = 0$ in $[-\pi, \pi]$.
You can tell from behavior of $\cos x$ that it doesn't equal 1 anywhere else in the interval.
The derivative exists everywhere, so 0 is the only critical point.

Example problem from test 2: $\sin x - x$ – page 2

Now let us look at derivatives at points before and after 0 (first derivative test) to check if it is local maximum or minimum.
$x < 0 \implies f'(x) < 0$ because $\cos x < 1$ there. So f is decreasing there.
$x > 0 \implies f'(x) < 0$ also for same reason. So f continues to decrease. So 0 is neither a local maximum nor a local minimum.

$\sin x - x$ - concavity and inflexion point

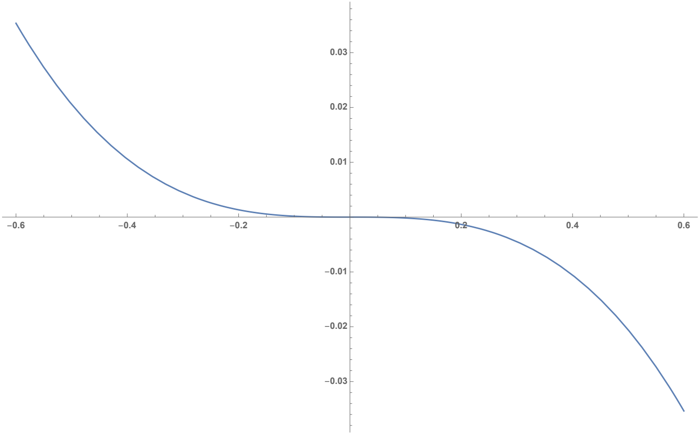
$f''(x) = -\sin x = 0$ at $x = 0$ and so it could be an inflexion point, but second derivative test is inconclusive.
$-\sin x$ goes from positive to negative as we go from negative side to positive side of the interval.
So the graph goes from concave up to concave down and at 0 it changes concavity because $f''(x)$ goes from positive to negative.
So 0 is indeed an inflexion point.

Note on extrema and inflexion point

NOTE: *Just because* $f'(x) = 0$ *doesn't mean peak or valley (local max / min) and just because* $f''(x) = 0$ *doesn't mean inflexion point.*
Example: $f(x) = x^4$ has $f''(x) = 12x^2 = 0$ at $x = 0$ but because $f''(x)$ doesn't change sign it doesn't change concavity, and so there is no inflexion point at 0.

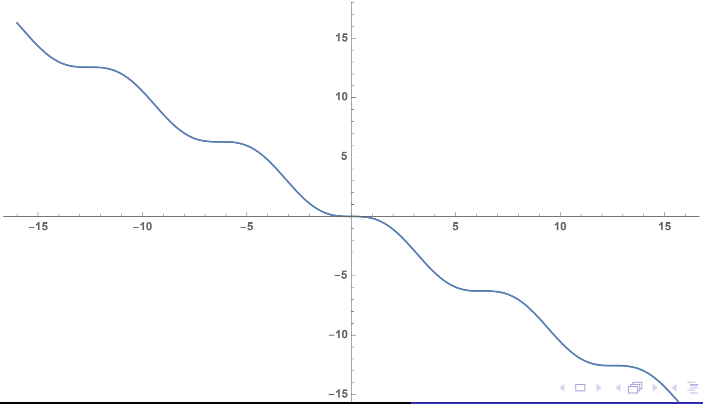
Graph for $\sin x - x$

0 is an "inflexion point"



Graph for $\sin x - x$ for all x

Second derivative $-\sin x = 0$ is 0 at all multiples of $\pi = 3.14159....$
So it has inflexion points at all those values.
Note how concavity also changes at those points.



Graph for $\sin x - x$ as $x \to +\infty$ or $x \to -\infty$

What happens to $\sin x - x$ as $x \to +\infty$ or $x \to -\infty$?
As $x \to +\infty$ or $x \to -\infty$ we see that $\sin x - x$ behaves more or less like the graph of the line $y = -x$. In other words, $y = -x$ is a **"slant asymptote"** for $\sin x - x$. (Not really an asymptote because the curve crosses the line).
In fact, you can show this is the case by looking at the limit

$$\lim_{x \to \pm\infty} \frac{\sin x - x}{-x}.$$

As $x \to +\infty$ or $x \to -\infty$ we see that

$$\lim_{x \to \pm\infty} \frac{\sin x - x}{-x} = \lim_{x \to \pm\infty} \frac{\sin x}{-x} + 1 = 1.$$

But saying that $f(x)/g(x) \to 1$ is same as saying $f(x) \to g(x)$.
NOTE: You could have found this limit using L'Hospital's rule as well – both numerator and denominator are going to infinity.

PROBLEM FROM SPRING 2010 TEST 3

Find the intercepts and asymptotes of $y = x - \dfrac{1}{x}$.

Using derivatives alone, say where $x - \dfrac{1}{x}$ is increasing or decreasing, concave up or down.

Also find the asymptotes, if any.

Graph the curve using the information above.

Graphing $x - (1/x)$ –domain, intercepts, asymptotes

DOMAIN:
All real numbers EXCEPT 0.

INTERCEPTS:
There is no $y-$intercept and in fact the $y-$axis is a vertical asymptote because $y(0) = 0 - \frac{1}{0}$ is undefined.
The $x-$intercepts are given by $x - (1/x) = 0$ which gives $x^2 - 1 = 0$ and thus $x = 1, -1$.
So the graph crosses the $x-$axis at 1 and -1.

ASYMPTOTES:
As it goes to ∞ or $-\infty$ the $1/x$ term goes to 0
and so the function is asymptotic to the graph of $y = x$,
i.e, asymptotic to the diagonal line $y = x$.

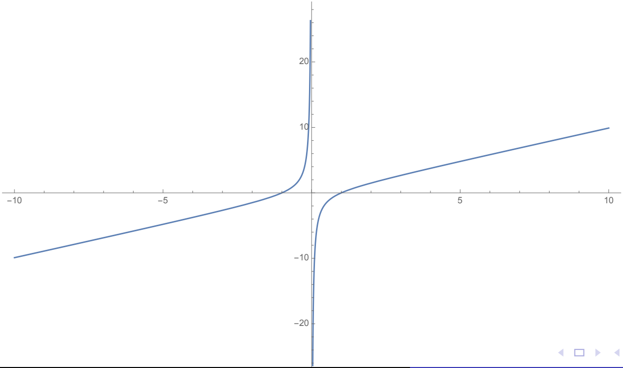
Graphing $x - (1/x)$ – critical points

The first derivative is $1 + \frac{1}{x^2}$
and it is always positive, never zero.
So no local maxima or minima.
0 is a critical point because derivative is undefined but $y \to \infty$ as $x \to 0$.
So the function is always increasing.

The second derivative is $-2/x^3$.
This is NEVER zero!
This is positive for negative values of x and negative for positive values.
[You can check this also by plugging in one negative value and one positive value].
So it is concave up on the negative side and concave down on the positive side.

Graphing $x - (1/x)$ -shape of graph

GRAPH: So the graph *looks like* a hyperbola especially near $x = 0$. There x is small and so it behaves like $y = -1/x$. One arm of the hyperbola is stretched and goes asymptotic to the $y-$axis on either side of the $y-$axis. The other arms go asymptotic to the diagonal line $y = x$.



4.6.11: FIND MINIMUM VALUE OF $x^2 \ln x$ USING CALCULUS

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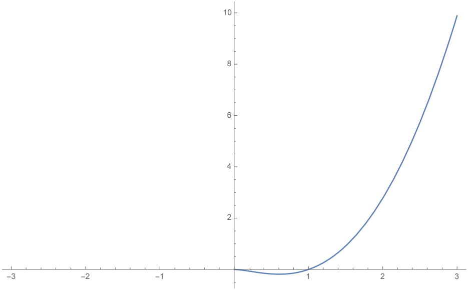

First note that $\ln x$ goes to $-\infty$ as $x \to 0^+$.
It is not even defined for $x \leq 0$.
The point here is that x is so much smaller than $\ln x$ near 0 that $x^2 \ln x$ goes to a finite number even though $\ln x \to -\infty$. (same will be true for $x \ln x$ also).

4.6.11: FIND MINIMUM VALUE OF $x^2 \ln x$ USING CALCULUS -contd

As $x \to 0^+$ using L'Hospital's rule (check conditions!)
we see that $\dfrac{\ln x}{1/x^2} \to \dfrac{1/x}{-2/x^3} = -x^2/2 \to 0$.
Differentiating using product rule,
$f'(x) = 2x \ln x + x^2(1/x) = 2x \ln x + x = x(2 \ln x + 1)$.
$f'(x) = 0 \implies x = 0 \text{ or } 2 \ln x + 1 = 0$
$2 \ln x + 1 = 0 \implies \ln x = -1/2 \implies x = e^{-1/2} = 0.61$
approximately.
So the critical points are 0 and 0.61.
NOTE: To see that at 0 the derivative is 0, you need to show that $f'(x) = 2x \ln x + x = 0$.
For this we need that $x \ln x = 0$ and this can be proved by L'Hospital's rule just as we showed $x^2 \ln x \to 0$.

4.6.11: $x^2 \ln x$ - putting it all together

Differentiating again using product rule,
$f''(x) = 2x(1/x) + 2(\ln x) + 1 = 2\ln x + 3.$
$f''(x) = 0 \implies x = e^{-3/2} = 0.22.$
Putting it all together, we get that the function is concave down from 0 to about 0.22, then concave up, (check some values before and after 0.22 !)
has a local minimum at about 0.61
and after that keeps increasing.
The local minimum is also absolute because it is the only minimum and it is decreasing before and increasing after, for all x.
GRAPHING THE FUNCTION (Not required for this problem, but just FYI): The x-intercepts are 0 and 1 because $x^2 \ln x = 0 \implies x = 0$ or $\ln x = 0$. As $x \to \infty$ we have $x^2 \ln x \to \infty$ because both x^2 and $\ln x$ go to infinity.