11-12-2020 Notes, Calculus 1

Definite & Indefinite Integrals, Fundamental Theorems (5.1,5.2,5.3,5.4)

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Outline

1. Antiderivative and Area
2. Summary of Integration
3. Definite Integral to find Area, Distance, etc.,
4. Riemann Sum (Precise definition of integral) ; Examples
5. Thanks, Fundamental Theorem of Calculus!
6. Properties of definite integrals
7. Fundamental Theorem –why it works ; more applications

Antiderivative and Area

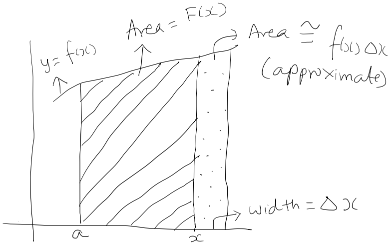
We defined antiderivative of $f(x)$ as the function $F(x)$ such that $F'(x) = f(x)$.

We will see that this is related to area under the curve that is the graph of $y = f(x)$ from $x = a$ to $x = b$.

Area function and its derivative

Think of the area of the big shaded area, under graph of $y = f(x)$ from a to x, **as a function by itself.**

What happens when we increase the interval from x to $x + \Delta x$?



Derivative of area function = antiderivative!

We will show (sketchily) that the

Derivative of Area Function = Value of function!!

Note that the area of dotted region is approximately $f(x)$ times Δx.

$$\lim_{\Delta x \to 0} \frac{F(x+\Delta x) - F(x)}{\Delta x} \simeq \lim_{\Delta x \to 0} \frac{f(x)\Delta x}{\Delta x} \to f(x).$$

The C in the antiderivative

What happens if we change the starting point from a to some other value, say b to x?

Then we get $F(x) + C$ where C is area from b to a (could be positive or negative).

Summary of what we know about Integral

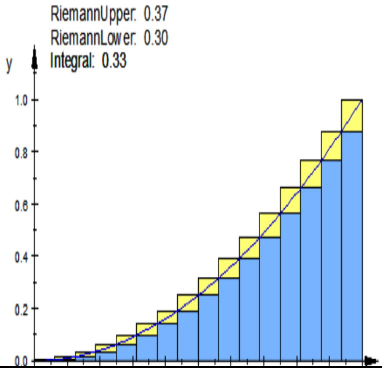
1. Area under graph (or "Sum of Values of function") can be approximated using rectangles.

2. As number of rectangles goes to infinity (width of each rectangle goes to zero) the approximation gets more precise. The limit of the sum gives exact area (if it exists).

3. The same sum can be found using antiderivative of function, under some conditions. This is guaranteed by the **Fundamental theorem of calculus.**

Example: area under parabola

Rectangles to calculate area under $y = x^2$ from $x = 0$ to $x = 1$.
(Actual value = 1/3 = 0.33).



Example: area under parabola – explanation

Remember that the integral can be thought of as a way to sum the values, and since there are infinitely many, we think of it as an area.

For each block you use one value from the graph as the height.

Example: area under parabola – explanation – 2

If you use left endpoints you get lower rectangles (blue).
If you use right endpoints you get upper triangles (yellow).
Area under graph lies between the two types of rectangles.
This area under the curve $y = x^2$ (or “sum of values of function x^2") is the **definite integral** $\int_a^b x^2 \, dx$

Example: Velocity and distance

PROBLEM FROM QUIZ 9, 2017
The speedometer of a car at Daytona speedway records the following velocities (see bottom of page) during a 1 minute period. Question in next slide.

Time(s)	Velocity (mph)
0	182.9
10	168.0
20	106.6
30	99.8
40	124.5
50	176.1
60	175.6

Example: Velocity and distance –answer

Question: Estimate the distance traveled by this car during that period using the velocities at the beginning of the interval. Pay attention to the units! Your answer will be 2.383 miles.
Each interval has 10 seconds or 10/3600 hours.
You multiply the time in each interval by the velocity at the beginning and add.
Answer: $10/3600 \times 182.9 + 10/3600 \times 168.0 + 10/3600 \times 106.6 + 10/3600 \times 99.8 + 10/3600 \times 124.5 + 10/3600 \times 176.1$
$= (1/3600)(1829 + 1680 + 1066 + 998 + 1245 + 1761) = 8579/360 = 2.383$ miles.

Example: Velocity and distance – explanation

WHAT DOES THE AREA UNDER THE CURVE REPRESENT?

For one thing, the area of rectangles has same units as speed times time, so it should give distance traveled.
You can also say area function is antiderivative of given function.
What is antiderivative of velocity function?
Same as asking: Derivative of what function is velocity?
Same as asking: Rate of change of what is velocity?
In this case, since the velocity is always positive, we can say it is the rate of change of distance.

DEFINITION OF THE RIEMANN SUM

The following definition makes more precise the idea of "approximating area by adding areas of rectangles."

$$\int_a^b f(x)dx = \lim_{max\ \Delta x_i \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i \tag{1}$$

This is the most general definition of **a definite integral** and it is called the **Riemann sum.**

DEFINITION OF THE RIEMANN SUM – 2

The points x_i^* could be left endpoints, right endpoints or midpoints or points where function takes the maximum or minimum for the interval, or any other kind of point! Only thing is we take one point from each interval.
Also, the width need not be the same for each rectangle in this definition! But it includes the sums where the width is the same, as well.

RIEMANN SUM – in plain language

The dx is a symbol that stands for "small change in x."
The integral sign $\int_a^b$ stands for the whole process of adding the areas of the rectangles between a and b and then taking the limit.

> It means that we add the areas of rectangles
> with width Δx_i and height $f(x_i^*)$
> and then keep increasing the number of rectangles (which is same as letting width get smaller and smaller) until we see a pattern and we can find the limit (IF IT EXISTS!).

Riemann Sum – when would limit exist?

For what sorts of functions would the limit in definition (1) above are guaranteed to exist?

Answer: Functions that are continuous or have only a finite number of jump discontinuities.
There are other more general *integrable* functions as well as more general *definitions* of integrals.

NOTE THAT THIS IS THE EXACT VALUE OF THE INTEGRAL!

IF YOU STOP WITH SOME FINITE NUMBER OF RECTANGLES THEN YOU GET AN APPROXIMATION.

Area under e^x using rectangles

Find the area under $y = e^x$ from $x = 0$ to $x = 1$.

Answer: This is given by $\int_0^1 e^x \, dx$.

NOTE: $\int f(x)\, dx$ means "antiderivative of $f(x)$." This is the **indefinite** integral. $\int_a^b f(x)\, dx$ is the **definite integral** as described in previous pages.

Area under e^x using rectangles – 2

Just for an example, we can divide the interval [0,1] into ten intervals each of width 0.1. Then just pick a number from each interval, say the midpoint.
Midpoint of [0,0.1] is 0.05, midpoint of [0.1,0.2] is 0.15, etc.,
You need to find the y-value at each of these mid-points and multiply by the width 0.1 to get the areas of the rectangles.
We then add the areas of these rectangles.

Area under e^x using rectangles – cont.d

Then the **approximation** is:

$$\int_0^1 e^x dx \sim (0.1)e^{0.05} + (0.1)e^{0.15} + ... + (0.1)e^{0.85} + (0.1)e^{0.95}$$

$$= (0.1)[e^{0.05} + e^{0.15} + e^{0.25} + e^{0.35}$$

$$+e^{0.45} + e^{0.55} + e^{0.65} + e^{0.75} + e^{0.85} + e^{0.95}]$$

$$= (0.1)\sum_{i=0}^{9} e^{0.05+i(0.1)} = 1.71756$$

EASIER WAY: USING ANTIDERIVATIVES

The fundamental theorem of calculus (part 2) says that the result we got above after a lot of difficult calculations using sums of areas of rectangles can be found using antiderivatives.

If f is continuous on [a,b] and $F'(x) = f(x)$ then

$$\int_a^b f(x)\, dx = F(b) - F(a).$$

That is, integral of the derivative of F equals $F(b) - F(a)$
Equivalently, it says “integral of function can be found using its antiderivative.”

We’ll see later how this amazing result comes about.

Applying the fundamental theorem

Thus $\int_0^\pi x^2\, dx = \frac{\pi^3}{3} - \frac{0^3}{3} = \pi^3/3$ because the derivative of $x^3/3$ equals x^2.

Note that you don't have to worry about the constant of integration. You can say that it cancels out when we subtract.

Similarly we can find $\int_0^1 e^x\, dx$.

It is exactly equal to change in antiderivative which is $e^1 - e^0 = 2.71828 - 1 = 1.71828$.

$$Find \int_0^{\pi/2} \sin t\, dt.$$

Note that this is the area under graph of $\sin t$ from 0 to $\pi/2$.

MORE APPLICATIONS OF FUNDAMENTAL THEOREM PART II

$$\int_0^{\pi/2} \sin t \, dt = [-\cos t]_0^{\pi/2} = (-\cos(\pi/2)) - (-\cos 0) = 1.$$

1. Find the area under $y = \frac{1}{x}$ from $x = 1$ to $x = 4$.

$$\int_1^4 \frac{dx}{x} = [ln x]_1^4 = ln 4 - ln 1 = ln 4 = 1.386.$$

MORE APPLICATIONS –sales problem

2. The monthly sales at a store are 10,000 dollars when it opened and declining at a rate $S'(t) = -10t^{2/5}$ dollars per month.
a) Find the formula for $S(t)$ after t months by integrating $S'(t)$.
b) Find $S(2)$.

MORE APPLICATIONS –sales problem – 2

We have $\dfrac{dS}{dt} = -10t^{2/5}$ with $S(0) = 10000$.
FTC II says: Total change from 0 to 2 equals integral from 0 to 2 of derivative.

$$S(2) - S(0) = S(2) - 10000 = \int_0^2 -10t^{2/5}dt$$

$$= \left[-10\frac{t^{(2/5)+1}}{(2/5)+1}\right]_0^2 = \left[-(50/7)t^{7/5}\right]_0^2 = -(50/7)(2^{7/5}) - (0)$$

Therefore $S(2) = 10000 - (50/7)(2^{7/5}) = 9981.15$ dollars.

Alternate solution of the sales problem

This can be solved also like this:

We have $\dfrac{dS}{dt} = -10t^{2/5}$ with $S(0) = 10000$.
Solving this differential equation (that is, writing the antiderivative), we get

$$S(t) = \int -10t^{2/5}dt = -10\frac{t^{(2/5)+1}}{(2/5)+1} + C = -(50/7)t^{7/5} + C.$$

NOTE: $\int f(x)\, dx$ (without the limits) just means the general antiderivative!
Setting $S(0) = 10000$ and solving for C, we get $C = 10000$.
So $S(t) = 10000 - (50/7)t^{7/5}$.
$S(2) = 10000 - (50/7)(2^{7/5}) = 9981.15$ dollars.

Velocity problem using fundamental theorem

Given that the velocity at time t of a particle is $v(t) = e^{-t}$ find:
(a) distance and displacement from $t = 0$ to $t = 10$ seconds
(b) acceleration at t seconds
(c) displacement at time t seconds if it starts from $s(0) = 0$.

Velocity problem using fundamental theorem – answer

Soln: The displacement

$$\int_0^{10} e^{-t}dt = \left[-e^{-t}\right]_0^{10}$$

$= -e^{-10} - (-e^0) = 1 - 0.000045 = 0.999955$. The distance is equal to the displacement in this case!

Actually it will be total distance as well. Why?

Answer: here the velocity is always positive.

Velocity problem using fundamental theorem – answer – 2

The acceleration at t is $v'(t) = -e^{-t}$.
The displacement $s(t)$ at t is the integral of velocity which equals e^{-t}
$s(t) = \int e^{-t}\, dt = -e^{-t} + C$.
To get C put $t = 0$ in this. We get $s(0) = 0 = -e^{0} + C$.
Solving, we get $C = 1$. So $s(t) = 1 - e^{-t}$.

PROPERTIES OF DEFINITE INTEGRAL -1,2,3

In all that is below, c is a constant.

1. $\displaystyle\int_a^b c\, dx = c(b-a)$

For example, if $c = 1$ then $\displaystyle\int_a^b 1\, dx = \int_a^b dx = (b-a)$.
This is just the area under $y = 1$ from $x = a$ to $x = b$.
The area is just height times width = $1 \times (b-a) = b - a$.

PROPERTIES OF DEFINITE INTEGRAL -1,2,3 –cont.d

2. $\int_a^b [f(x) \pm g(x)]\, dx = \int_a^b f(x)\, dx \pm \int_a^b g(x)\, dx$

Just as with derivatives and antiderivatives, the definite integral of a sum (or difference) can be found by finding the definite integrals separately and then adding (or subtracting) them.
In terms of areas (or adding y-values) you see that whether we add the areas (or y-values) for different x-values before or after adding the y-values we will get same result.

PROPERTIES OF DEFINITE INTEGRAL -1,2,3 –cont.d

3. $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$

Just as with derivatives and antiderivatives, when you multiply a constant or a fixed number it just tags along. Definite integral can be found by finding the definite integral separately and then multiplying by the constant later.
In terms of areas (or adding y-values) you see that whether we add the areas (or y-values) for different x-values before or after multiplying the y-values by the constant we will get same result.

PROPERTIES OF DEFINITE INTEGRAL -4,5,6

Properties 4,5,6,7 are peculiar to definite integrals. You don't have them for derivatives or antiderivatives. $\forall x \in [a, b]$ means “for all x in the interval $[a, b]$."

4. $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

Basically we are saying that you can combine the areas over two different intervals on the x-axis.

PROPERTIES OF DEFINITE INTEGRAL -4,5,6 –cont.d

Properties 4,5,6,7 are peculiar to definite integrals. You don't have them for derivatives or antiderivatives. $\forall x \in [a, b]$ means "for all x in the interval $[a, b]$."

5. $f(x) \geq 0, \forall x \in [a, b] \implies \displaystyle\int_a^b f(x)\, dx \geq 0.$
6. $f(x) \geq g(x), \forall x \in [a, b] \implies \int_a^b f(x)\, dx \geq \int_a^b g(x)\, dx.$

PROPERTIES OF DEFINITE INTEGRAL -4,5,6 –cont.d

7.

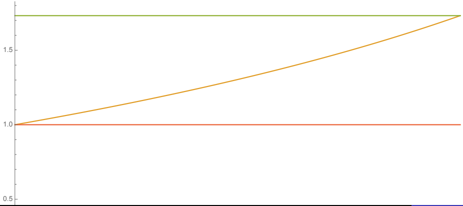
$$m \leq f(x) \leq M \,\forall x \in [a, b] \implies m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a).$$

These properties talk about how you can approximate the area if you know that the values are always within certain bounds. NOTE: INTEGRAL COULD BE NEGATIVE, BUT AREA IS ALWAYS POSITIVE!

Application of property 7

Problem 5.2.61: Using Property 7, estimate $\int_{\pi/4}^{\pi/3} \tan x \, dx$

To use property 7, we need lower and upper bounds for $\tan x$ in all of the interval $[\pi/4, \pi/3]$.
As shown in the graph below, $\tan x$ is nicely bounded below by $\tan(\pi/4) = 1$ and $\tan(\pi/3) = \sqrt{3} \simeq 1.732$ in that interval.
It helps that $\tan x$ is a nice increasing function in this interval.



Application of property 7 -answer

Property 7 says $m \le f(x) \le M, \forall x \in [a,b]$

$\implies m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$.

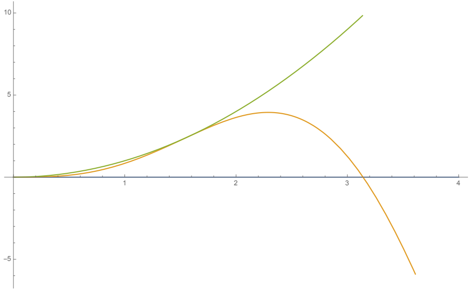
Applying this get $1 \le \tan(x) \le \sqrt{3}, \forall x \in [\pi/4, \pi/3]$

$$\implies 1\left(\frac{\pi}{3} - \frac{\pi}{4}\right) \le \int_{\pi/4}^{\pi/3} \tan x\,dx \le \sqrt{3}\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$$

$$\implies \frac{\pi}{12} \le \int_{\pi/4}^{\pi/3} \tan x\,dx \le \frac{\sqrt{3}\pi}{12}$$

EXAMPLE FOR USING PROPERTY 6

Using Property 6, estimate $\int_0^{\pi} x^2 \sin x \, dx$.



The area from 0 to π is the area above the $x-$axis. The brown line is the graph of $x^2 \sin x$. The green line is the graph of x^2.

EXAMPLE FOR USING PROPERTY 6 –answer

Since $\sin x \leq 1$ always, $x^2 \sin x \leq x^2$ always.
So by Property 6, $\int_0^\pi x^2 \sin x \, dx \leq \int_0^\pi x^2 \, dx$.

(This basically says that area under green line will be bigger than area under brown line)

Part I of fundamental theorem (FTC)

$$\int_0^x f(t)\,dt = g(x) \implies g'(x) = f(x).$$

If function is continuous, it can be integrated,
and the derivative of its integral equals the function itself.

$$\int_0^x f(t)\,dt = g(x) \implies g'(x) = f(x).$$

So the integral and derivative are kind of inverse of each other. I say “kind of” because the antiderivative is not just one function, but a set of functions of the form $F(x) + C$.

What part I of FTC really says

What part I says is that the small change in area
(which looks like $f(x)\Delta x$)
divided by small change in x, namely Δx,
approaches $f(x)$.

$$\text{But Limit of } \frac{\text{change in area function}}{\text{change in } x}$$

equals derivative of area function!

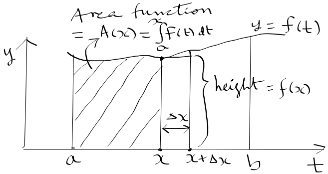
What Part II of FTC really says

Remember, Part II says: If $F'(x) = f(x)$ (i.e, F is the antiderivative of f) then $\int_a^b f(x)\,dx = F(b) - F(a)$.

What does it really mean? A “Hand Waving” explanation:
If you think of the area under the graph of $f(x)$ as a function by itself, then a small change in the area is given by $f(x)\Delta x$.
Remember, the area is calculated using the antiderivative, so $f(x)$ is also the derivative of the area function.

> So what part II of the theorem says is that
> *when you add up the small changes between a and b*
> *you get the total change in area from a to b.*

A picture to illustrate the FTC



Small change in area $= \Delta A$
$\simeq$ Area of vertical strip of height $f(x)$ and width Δx

$\Delta A \simeq f(x)\,\Delta x$ (approximately!)

FTC II: Sum of small changes = Total change

$$\int_a^b f(t)\,dt = \lim_{\Delta x \to 0} \sum (\Delta A) = A(b) - A(a)$$

FTC I: Derivative of $A(x) = f(x)$

$$\lim_{\Delta x \to 0} \frac{\Delta A}{\Delta x} \to f(x)$$

$\left(\hookrightarrow \simeq \frac{f(x)\Delta x}{\Delta x} \right)$

More explanation of picture

The shaded area is the area function (we think of area under graph of $f(x)$ as a function).
A small change in area function is given by the vertical strip of width Δx and height (from its left endpoint) $f(x)$.
So its area is *approximately* $f(x)\Delta x$.
Note that this is only approximate!
You can see in picture that area under curve in vertical strip is not exactly area of rectangle of height $f(x)$.
So we see that as $\Delta x \to 0$, the limit of $\Delta A/\Delta x \to f(x)$.
But this limit is exactly same as $A'(x)$.
So derivative of area function $A(x)$ gives $f(x)$ and thus $A(x)$ is an antiderivative of $f(x)$.

PROBLEM 5.3.11

$g(x) = \int_x^0 \sqrt{1+\sec t}\, dt. \quad g'(x) = ?$

Note: The problem should say that $x \in [0, \pi/2)$ *because* $\sec x$ *is not continuous at* $\pi/2$.

Use the following PROPERTY OF INTEGRAL: (Not mentioned earlier!) $\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx \implies g(x) = \int_x^0 \sqrt{1+\sec t}\, dt = -\int_0^x \sqrt{1+\sec t}\, dt = \int_0^x -\sqrt{1+\sec t}\, dt.$

Then by Part I of fundamental theorem, $g'(x) = -\sqrt{1+\sec x}$.

A Slightly More Complicated Application of FTC I

PROBLEM 5.3.61: $F(x) = \int_x^{x^2} e^{t^2}\, dt. \quad F'(x) = ?$

By subtracting areas, and calling area from 0 to u as $G(u)$, we see that

$$\int_x^{x^2} e^{t^2}\, dt = \int_0^{x^2} e^{t^2}\, dt - \int_0^{x} e^{t^2}\, dt = G(x^2) - G(x)$$

Now differentiating both sides and using Part I of fundamental theorem,

$$F'(x) = G'(x^2)(2x) - G'(x) = 2xe^{(x^2)^2} - e^{x^2} = 2xe^{x^4} - e^{x^2}.$$

$G(x^2)$ is differentiated using chain rule.