11-11-2020 Notes, Calculus 1

Introduction to Integrals
(4.9) Antiderivatives

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Outline

Antiderivative – Inverse of Derivative

Finding functions with specified derivatives (rates of change)

Basically, we will try to remember what function resulted in a given derivative

The **antiderivative of $\mathbf{f}(\mathbf{x})$ will be denoted by $\mathbf{F}(\mathbf{x})$**.
So we have $\mathbf{F}'(\mathbf{x}) = \mathbf{f}(\mathbf{x})$.

EXAMPLES:
$(x^2)' = 2x$, so if $f(x) = 2x$, then $F(x) = x^2$.
What if $f(x) = x$?

Antiderivative of x^n

If $f(x) = x$ then $F(x) = x^2/2$ so that
$F'(x) = (x^2/2)' = (1/2)(x^2)' = (1/2)(2x) = x$.

If $f(x) = x^2$, then you can guess that $F(x) = x^3/3$.

In general,

$$\textbf{antiderivative of } \mathbf{x^n} = \frac{\mathbf{x^{n+1}}}{\mathbf{n+1}}.$$

WARNING: This formula DOES NOT WORK FOR n = -1!!
You get 1/0 which is meaningless.
For $\mathbf{x^{-1} = 1/x}$ we know that antiderivative is $\ln \mathbf{x}$.

Antiderivative of kx^n

What is the antiderivative of kx^n? (k is a fixed constant).
Remember that when you differentiate, k just tags along.
So $(kx^2/2)' = k(x^2/2)' = kx$. (NO NEED FOR PRODUCT RULE !)
So $F(x) = kx^2/2$ is an antiderivative of kx.
In general $kF(x)$ is an antiderivative of $kf(x)$.
So $k \sin x$ is an antiderivative of $k \cos x$,
ke^x is an antiderivative of ke^x, and so on.
In other words, **antiderivative of constant times a function is the constant times the antiderivative**

Antiderivative of e^{kx}

What about e^{kx}?

Again, look at what happens when you differentiate.
$(e^{kx})' = ke^{kx}$, so we need to divide by k.

$$\left(\frac{e^{kx}}{k}\right)' = \frac{(e^{kx})'}{k} = (ke^{kx})/k = e^{kx}.$$

$\mathbf{F(x) = e^{kx}/k}$ **is an antiderivative of** $\mathbf{e^{kx}}$.

$F(x) + G(x)$ and an example

Also when you differentiate $F(x) + G(x)$, you just add $F'(x)$ and $G'(x)$. So you can guess that if $f(x) = 1 + 2x$ then $x + x^2$ will give $1 + 2x$.

EXAMPLE: Find an antiderivative of $3x^4 + 5\sqrt{x} - 2e^{-2x}$

Antiderivative of $3x^5$ is $3\left(\dfrac{x^{4+1}}{4+1}\right) = 3\left(\dfrac{x^5}{5}\right) = \dfrac{3x^5}{5}$.

Antiderivative of $5\sqrt{x} = 5x^{1/2}$ is 5 times antiderivative of $x^{1/2}$:

$5\dfrac{x^{(1/2)+1}}{(1/2)+1} = 5\dfrac{x^{3/2}}{3/2} = \dfrac{10}{3}x^{3/2}$.

Antiderivative of $-2e^{-2x}$ is $-2\left(\dfrac{e^{-2x}}{-2}\right) = e^{-2x}$.

Finally, add all : $F(x) = \frac{3x^5}{5} + \frac{10}{3}x^{3/2} + e^{-2x}$.

THE MOST GENERAL ANTIDERIVATIVE

There can be more than one antiderivative !

For example, $x^2+1, x^2+2, x^2+3.1415, x^2+\sqrt{2}, \ldots$ ALL give $2x$ when differentiated.

To express all the possibilities, we say that
THE antiderivative of 2x is $\mathbf{x^2+C}$ where C can be any fixed number.

Geometrically, we are saying that different curves can have the same slope of tangents, i.e, parallel tangents

Example: Find the most general antiderivative of $e^2+\cos x+\sqrt{x}$.

Problem from Quiz 10, Fall 18

Find the most general antiderivative of $e^2 + \cos x + \sqrt{x}$.
Answer: Antiderivative of e^2 is $e^2 x$ because e^2 is just a number.
Antiderivative of $\cos x$ is $\sin x$.
Antiderivative of $\sqrt{x}$ is obtained by writing $\sqrt{x} = x^{1/2}$.
It is $\dfrac{x^{(1/2)+1}}{(1/2)+1} = \dfrac{x^{3/2}}{3/2} = \dfrac{2}{3}x^{3/2}$.
Putting everything together we get that

$$\int (e^2 + \cos x + \sqrt{x})dx = e^2 x + \sin x + \frac{2}{3}x^{3/2} + C.$$

The integral sign $\int$ just says that we are finding the antiderivative.

Application to velocity and distance

PROBLEM FROM QUIZ 8, SPRING 2016.

A particle has velocity $\pi \sin t$ after t seconds. Find its position function $s(t)$ using anti-derivatives if its starting position is given by $s(0) = 0$.

Antiderivative of velocity function is the position function, because derivative of position function is the velocity function.
So among all the curves with slope given by $\pi \sin t$ we want the curve passing through (0,0).
Antiderivative of $\pi \sin t = -\pi \cos t + C$. So $s(t) = -\pi \cos t + C$.
Given $s(0) = 0$ we get
$-\pi \cos 0 + C = 0 \implies -\pi + C = 0 \implies C = \pi$.
So the position function is $s(t) = -\pi \cos t + \pi = \pi(1 - \cos t)$.

PROBLEM 4.9.33

Find f if $f'(t) = 4/(1+t^2)$ and $f(1) = 0$.

Solution: We have $(\tan^{-1} x)' = \dfrac{1}{1+x^2}$.
So $4(\tan^{-1} x)' = \dfrac{4}{1+x^2}$.
So $f(t) = 4\tan^{-1} x + C$.
From $f(1) = 0$ we get
$4\tan^{-1}(1) + C = 0 \implies 4(\pi/4) + C = 0 \implies C = -\pi$.
So the answer is $f(t) = 4\tan^{-1} x - \pi$.