10-21-2020 Notes, Calculus 1

Maxima / Minima using Derivative

Sankar Sitaraman

Math Dept, Howard University

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Outline

1. Rest of semester
2. Finding Maxima and Minima : Basic ideas
   - Examples
3. Solution to some problems from 3.9,3.10, 4.1

Rest of semester

ONLY ABOUT A MONTH TO GO!
PLEASE DO ALL REMAINING QUIZZES
PREPARE WELL FOR QUIZZES

Main Idea

Maximum or minimum of a function occurs
where graph has peak or valley

At points where function has peaks and valleys
(aka critical values)
Either derivative is ZERO
or
derivative is UNDEFINED

Can you think of a function with a peak or valley where it's derivative doesn't exist?

Local and Global Extrema

Also, the peaks and valleys may be *local* maxima or minima or they may be *global or absolute* maxima or minima.

NOTE: "Extremum" means Either maximum or minimum

FACT: Absolute minimum and maximum WILL exist IF function is CONTINUOUS on a CLOSED interval.
In the following pages you will see that if function is *not continuous* then absolute maximum may not exist.

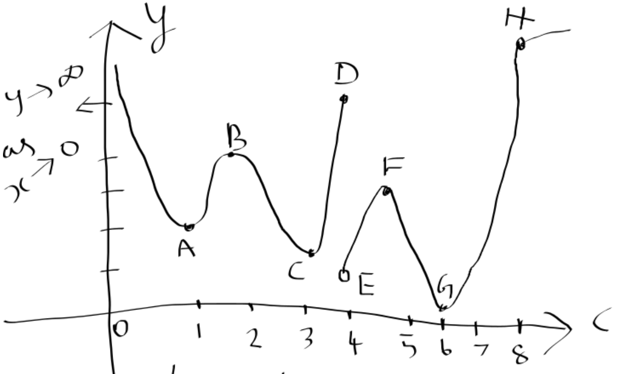
When can derivative be zero?

Although derivative is usually zero at peaks and valleys, it can be zero at other places, too.

Examples: A constant function $y = c$ has derivative equal to 0 always.
A function like $y = x^3$ has derivative zero at the "saddle point."

Moral of the story:
Just because you found a place where derivative is zero, doesn't mean you found a maximum or minimum value.
But we know (Fermat's theorem says so) that
IF THERE IS A PEAK OR VALLEY, DERIVATIVE IS ZERO THERE, PROVIDED IT EXISTS

EXAMPLE GRAPH WITH PEAKS AND VALLEYS



$y' = 0$ at $x = 1, 2, 3, 5, 6$

A, B, C, D, F, G, are local extrema

G is also a global minimum

H is the absolute (global) maximum

In the interval

EXAMPLE GRAPH WITH PEAKS AND VALLEYS – ANSWERS

The critical numbers are $x = 1, 2, 3, 4, 5$ and 6.
$x = 8$ is NOT a critical number because derivative is neither zero nor undefined there.
The absolute minimum *value* of $y = f(x)$ is at $x = 6$ and it is about 0.5.
There is no absolute maximum for this graph in [0,8] because the function is going to infinity near zero but if you take any subinterval [m,8] where m is not 0 but less than 8 then you can get an absolute maximum.
This doesn't contradict the FACT that a continuous function will have an absolute maximum and minimum in a closed interval because the function is not continuous at 0 and also at 4.

GENERAL PROCEDURE TO FIND EXTREMA -STEP 1

STEP 1:
First **find critical points** by looking at where $f'(x) = 0$ or $f'(x)$ DNE.
In other words, solve $f'(x) = 0$ for x or see where $f'(x)$ cannot be found.
Usually $f'(x)$ DNE if $f(x) \to \infty$ or is discontinuous or has a pointed edge.

GENERAL PROCEDURE TO FIND EXTREMA -STEP 2

STEP 2: Once you find x where $f'(x) = 0$
you can **check whether they are local maxima or minima** by looking at values to left and right and seeing whether function is increasing and then decreasing or decreasing and then increasing.
[Note, though, that you can find absolute max/min *without* finding whether a critical number is local max/min].

GENERAL PROCEDURE TO FIND EXTREMA -STEP 3

STEP 3:

To find absolute max/min,

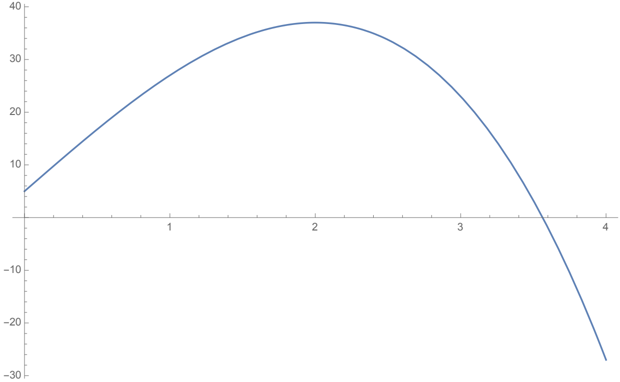
compare values of $f(x)$ at critical numbers

as well as boundary points

and see which is the highest and which is lowest.

Problem 4.1.48:

Find the absolute maximum and absolute minimum of $f(x) = 5 + 24x - 2x^3$ in [0,4].

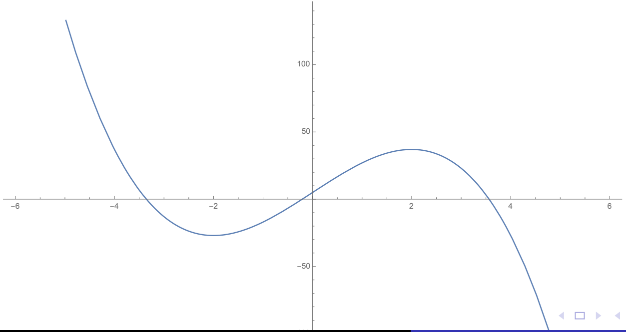


Answer to Problem 4.1.48

$f(x) = 5 + 24x - 2x^3 \implies f'(x) = 24 - 6x^2.$
$f'(x) = 0 \implies 24 - 6x^2 = 0 \implies 4 = x^2 \implies x = \pm 2.$
Only 2 is in [0,4].
$x < 2 \implies f'(x) > 0$ (f *increases*); $x > 2 \implies f'(x) < 0$ (f *decreases*).
So 2 is a local maximum.
$f'(x) = 24 - 6x^2$ is defined everywhere, and there is no point where it is undefined.
So 2 is the only critical point (points where derivative is 0 or undefined)
Now compare values at critical point AND boundary points.
$f(0) = 5, f(2) = 37, f(4) = -27.$
So 37 is absolute maximum and -27 is absolute minimum.

What happens over the entire real line?

What is you take the interval to be $(-\infty, \infty)$?
From graph below you see that it will have neither absolute maximum nor minimum. This shows the importance of the fact that in order for a continuous function to have an absolute maximum and minimum *we must be in a closed interval*. The entire real number line is an *open interval.*



Problem 4.1.55:

Find the absolute maximum and absolute minimum of

$$f(t) = t - \sqrt[3]{t}$$

in $[-1, 4]$.

Problem 4.1.55: Answer

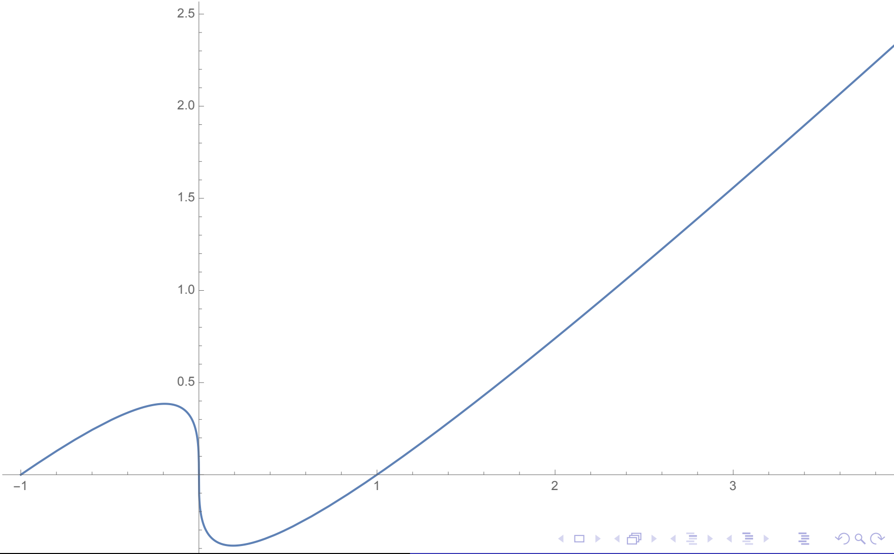
$$f(t) = t - \sqrt[3]{t} \implies f'(t) = (t - t^{1/3})' = 1 - \frac{1}{3}t^{-2/3}.$$

$$f'(t) = 0 \implies 1 - \frac{1}{3}t^{-2/3} = 0 \implies 1 = \frac{1}{3}t^{-2/3}$$

$$\implies 3 = t^{-2/3} \implies 27 = t^{-2} \implies t = \pm\sqrt{1/27} = \pm\frac{1}{3\sqrt{3}}.$$

Both $\frac{1}{3\sqrt{3}}$ and $-\frac{1}{3\sqrt{3}}$ are in $[-1, 4]$.
Also $f'(t)$ is undefined at $t = 0$ and $0 \in [-1, 4]$.
So we get three critical points totally.
Now we will try to see where it has local maxima, between $1/(3\sqrt{3})$ and $-1/(3\sqrt{3})$.

Problem 4.1.55: Graph



Problem 4.1.55: Conclusion

Check using derivative values before and after the two critical points that
$-\dfrac{1}{3\sqrt{3}}$ is a local maximum and $\dfrac{1}{3\sqrt{3}}$ is a local minimum.
Now compare values at critical point AND boundary points.

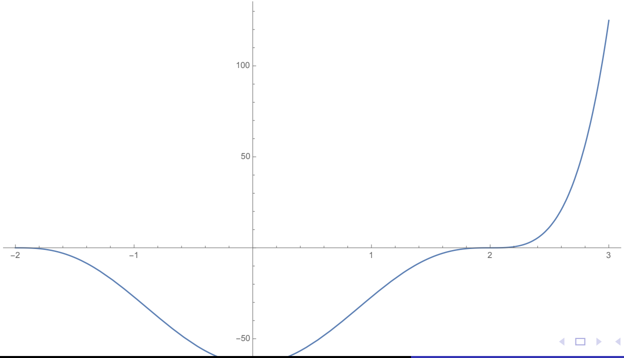
$$f(-1) = 0, f(-1/\sqrt{27}) = 0.385, f(1/\sqrt{27}) = -0.385,$$
$$\textit{and} \quad f(4) = 4 - 4^{1/3} = 2.412.$$

So at $x = 4$ we get absolute maximum is $f(4) = 2.412$
and at $x = 1/\sqrt{27} = -0.385$ is absolute minimum.

Solution to Problem 4.1.52:

Find the absolute maximum and minimum of $(t^2 - 4)^3 = 0$ in [-2,3].

NOTE: The Extreme Value Theorem (denoted by "FACT") guarantees that they will exist because $(t^2 - 4)^3$ is continuous and differentiable everywhere, so in particular in [-2,3].



Solution to Problem 4.1.52: critical points

First find critical points using the derivative f'.
Use chain rule to differentiate.

$$f'(t) = ((t^2-4)^3)' = 3(t^2-4)^2(t^2-4)' = 3(t^2-4)(2t).$$

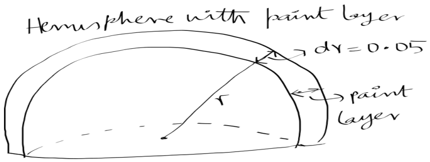
This derivative function is defined everywhere,
and it is zero when $t^2-4=0$ or $t=0$.
So $-2, 0, 2$ are the critical points.
Comparing values at the critical points and boundary points,
$f(-2)=0, f(0)=-64, f(2)=0, f(3)=125$
So absolute minimum is -64, at $x=0$,
and absolute maximum is 125, at $x=3$.

Solution to Problem 4.1.52: LOCAL maxima / minima

What are the local maxima, minima?
-2 and 2 are actually *inflexion points.*
Although the derivative is zero, they are neither maximum nor minimum.
In 4.3 we will see that this is because the second derivative is also zero here.
The graph changes from concave up to concave down or vice-versa.

Solution to problem 3.10.36

Problem 3.10.36: Use differentials to estimate the amount of paint needed to apply a coat of thickness 0.05cm thick to a hemispherical dome of diameter 50m.



Solution to problem 3.10.36 –solution

The volume of paint is just the volume of the layer of thickness $dr = 0.05$ shown in the picture.

This is the difference in the volume as the radius goes from $r = 25m = 2500cm$ to $r = 2500 + 0.05$ cms.

Solution to problem 3.10.36 –cont.d

The change in volume (=amount of paint) can be approximated by $dV = V'(r)dr$ where V is the Volume of hemisphere = $V = \frac{1}{2}\left(\frac{4}{3}\pi r^3\right) = \frac{2}{3}\pi r^3.$

$$dV = V'(r)dr \text{ (at } r = 2500, dr = 0.05) = V'(2500)(0.05)$$

$$= \frac{2}{3}\pi(3r^2)(dr) = 2\pi r^2 dr \text{ (at } r = 2500, dr = 0.05)$$

$= 2\pi(2500)^2(0.05) = 1963495.40$ cc approximately 518 gallons .

Solution to problem 3.9.33

3.9.33: The top of a ladder slides down a vertical wall at the rate of 0.15 m/s. At the moment when the bottom of the ladder is 3 m from the wall, it is sliding away from the wall at the rate of 0.2 m/s. How long is the wall?
Let y be height of top, x be distance of bottom from wall, l length.
Notice that the length is fixed!
$x^2+y^2=l^2 \implies \frac{d}{dt}(x^2+y^2)=\frac{d}{dt}(l^2) \implies 2x(x')+2y(y')=0.$
Simplifying, using $x=3, x'=dx/dt=0.2, dy/dt=-0.15$, we get $xx'+yy'=l(l') \implies 3(0.2)+y(-0.15)=0 \implies y=-0.6/(-0.15)=4$. From this we get
$l=\sqrt{x^2+y^2}=\sqrt{3^2+4^2}=5m.$