10-1-2020 Notes, Calculus 1

Derivatives of Trigonometric Functions

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Outline

The two main formulae

The main formulae to remember are

$$(\sin x)' = \cos x$$

$$(\cos x)' = -\sin x$$

Derivative of $\sin x$ using Limit formula

$$(\sin x)' = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sin(x+h)-\sin x}{h}$$

$$= \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$

$$= \lim_{h\to 0} \frac{\sin x(\cos h - 1) + \cos x \sin h}{h}$$

$$= \sin x \lim_{h\to 0} \frac{\cos h - 1}{h} + \cos x \lim_{h\to 0} \frac{\sin h}{h}$$

Why is this same as $\sin x \cos'(0) + \cos x \sin'(0)$?
Remember: $\sin 0 = 0, \cos 0 = 1.$

$\sin'(0) = 1$ means $\cos'(0) = 0$ and $(\sin x)' = \cos x$!

$$\cos'(0) = \lim_{h\to 0}\frac{(\cos h - 1)}{h} = \lim_{h\to 0}\frac{(\cos h - 1)}{h} \times \frac{\cos h + 1}{\cos h + 1}$$

$$= \lim_{h\to 0}\frac{-(\sin h)^2}{h} \times \frac{1}{\cos h + 1}$$

$$= \lim_{h\to 0}(-\sin h) \times \lim_{h\to 0}\frac{\sin h}{h} \times \lim_{h\to 0}\frac{1}{\cos h + 1}$$

$$= 0 \times \sin'(0) \times \frac{1}{2}$$

This is zero IF $\sin'(0)$ exists as a finite number.

Proof that $\sin'(0) = 1$

SO KEY PROBLEM IS TO FIND $\sin'(0) = \lim_{h \to 0} \frac{\sin h}{h}$

In fact, we will show that $\lim_{h \to 0} \frac{\sin h}{h} = 1.$

Then

$$(\sin x)' = \sin x \cos'(0) + \cos x \sin'(0) = \sin x(0) + \cos x(1) = \cos x.$$

Finding $\sin'(0)$ (limit of $\sin h/h$)

PROOF THAT $\lim_{h\to 0} \frac{\sin h}{h} = 1.$
In order to prove this, we can show that when h is small,

$$\cos h \leq \frac{h}{\sin h} \leq \frac{1}{\cos h}$$

Now if we use squeeze theorem we get $h/\sin h \to 1.$
As $h \to 0$, both $\cos h$ AND $1/\cos h \to 1.$
Therefore $h/\sin h \to 1$ also.
This in turn means $(\sin h)//h \to 1$ because, according to limit laws, if the limit of $f(x)$ is finite then limit of $1/f(x)$ is reciprocal of limit of $f(x)$.

PROOF THAT $\cos h \leq \dfrac{h}{\sin h} \leq \dfrac{1}{\cos h}$

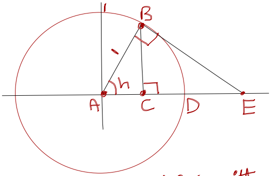
$$\lim_{h \to 0} \frac{\sin h}{h}$$

We will show that

$$\cos h \leq \frac{h}{\sin h} \leq \frac{1}{\cos h}$$

when h is small

Note: h is in radians



Compare areas of $\triangle ABC$ with those of sector ABD & $\triangle ABE$

$$\frac{\cos h \sin h}{2} \leq \frac{1^2 \times h}{2} \leq \frac{\tan h \times 1}{2}$$

PROOF THAT $\cos h \leq \frac{h}{\sin h} \leq \frac{1}{\cos h}$ – conclusion

Now divide all terms of the inequality by $\sin h/2$.
Then you get the desired inequality

$$\cos h \leq \frac{h}{\sin h} \leq \frac{1}{\cos h}$$

So we have shown that

$$(\sin x)' = \cos x$$

In a very similar way, we can show that

$$(\cos x)' = -\sin x.$$

PROOF THAT $\cos h \leq \frac{h}{\sin h} \leq \frac{1}{\cos h}$ – AN EASIER LOOK

Now, this is NOT a proper proof as the one we got above using areas of triangles, but perhaps easier to understand and helps us to see what is going on.

In the picture on page 7, look at the height BC of the triangle ABC. it just equals $\sin h$ because
$\sin h = (opposite)/(hypotenuse) = (length\ of\ BC)/1$.
Now look at the chord BD. It equals h because, by definition of radian angle, the length of BD equals radius times angle which is 1 times h.
Now, as $h \to 0$, you can see that the height BC almost merges with the chord BD because in a very microscopic neighborhood, the circle looks like a straight line. So as $h \to 0$, we see that $\sin h \to h$ and thus $\sin h/h \to 1$.

Derivative of $\tan x = \sin x / \cos x$

Use quotient rule:
$f'(x) = [(NUM)'DEN - (DEN)'NUM]/(DEN^2)$.

$$(\tan x)' = \frac{(\sin x)' \cos x - (\sin x)(\cos x)'}{(\cos x)^2}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x.$$

Similarly we can find the derivatives of *cotx*, *cosecx*, $\sec x$, *etc*.,