9/27/2018 Fall 2018,Proofs and Problem Solving I Sitaraman

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Midterm test review problems

Problems are mostly from "Discrete Mathematics" by Johnsonbaugh and Raji's Introduction to Number Theory ebook (linked on update page).

1. Use proof by contrapositive to show that for all real numbers x, if x^2 is irrational, then x is irrational. Is the converse true? Prove or give a counterexample.

   Solution: Contrapositive of $A \rightarrow B$ is $NOT\ B \rightarrow NOT\ A$.

   So contrapositive here is : If x is rational, then x^2 is rational.

   This we can prove directly:

   $$If\ x = k/m \in \mathbb{Q}\ then\ x^2 = k^2/m^2 \in \mathbb{Q}.$$

   Converse is false: $\sqrt{2}$ is irrational but $(\sqrt{2})^2 = 2$ is rational.

2. Show that the integer $Q_n = n! + 1$, where n is a positive integer, has a prime divisor greater than n. Conclude that there are infinitely many primes. Notice that this exercise is another proof of the infinitude of primes.

   Solution: Suppose there are no prime divisors of Q_n that are bigger than n. Now it is also true that thre are no the prime divisors of Q_n smaller than or equal to n. This is because if p is smaller than n then p divides $n!$ since $n!$ is just the product of all natural numbers smaller than n. Therefore Q_n has no divisors $\leq n$ or $> n$. This means it has no prime divisors. Now if Q_n has any divisors other than 1, then that divisor can be broken up into prime factors eventually and so this means that Q_n has no divisors other than 1 at all (including itself!). That means Q_n itself is 1. But clearly it is bigger than 1 !

3. Show that there are no prime triplets other than 3,5,7.

   Solution: Let $p, p+2, p+4$ be three numbers that form a triplet of the same form as 3,5,7, with $p > 3$.

   We will prove that 3 divides one of them, which would mean one of them is not a prime. Therefore there cannot be a prime triplet. The proof will be by cases.

   Case 1: If 3 divides p then we are done.

   Case 2: Assuming 3 does not divide p, we have two possibilities:

   EIther 3 divides $p+1$ or 3 divides $p+2$. (Among any three consecutive natural numbers, one will be a multiple of 3).

   If 3 divides $p+2$ we are done because then $p+2$ is not a prime. If not, then 3 divides $p+1$ and therefore we can say $p+1 = 3m$. Then $p+4 = p+1+3 = 3m+3 = 3(m+1)$ and so 3 divides $p+4$ we are done.

4. Show that if $2^n - 1$ is prime, then n is prime.

   Hint: Use the identity $a^{kl} - 1 = (a^k - 1)(a^{k(l-1)} + a^{k(l-2)} + ... + a^k + 1)$.

   Solution: You don't really need the full identity. All we need is that the RHS is a product of two numbers both bigger than 1 if $k > 1$.

   Proof by contradiction: Assume n is not a prime, and let $n = km$ with both k and m bigger than 1. Then using the identity we get $2^n - 1 = 2^{km} - 1 = (2^k - 1)M$ where M is an integer bigger than 1, equal to $1 + 2^k + 2^{2k} + ... + 2^{k(m-1)}$. On the other hand if $k > 1$ we have $2^k > 2$ and hence $2^k - 1 > 1$. Thus $2^n - 1$ is a product of two numbers both bigger than 1 and it is not a prime, contradicting the assumption.

5. Recall that we proved that gcd of two numbers m, n is the smallest positive integer d such that $d = xm + yn$ for some integers x, y. Find the gcd of 2 and 13 using Euclidean algorithm. Show that gcd of m, n divides any number of the form $xm + yn$. Either find all solutions or prove that there are no solutions for the diophantine equation $2x + 13y = 31$. (i.e, x, y that satisfy the equation cannot be integers).

   In one of the homework problems we showed that gcd of m, n divides any number of the form $xm + yn$.

   So the trick here is to write the gcd in the form $2x + 13y$ and then if 31 is a multiple of the gcd we can get the solutions by multiplying both sides by suitable factor.

   To write gcd in this form we use Euclidean algorithm.

   In this case it takes only one step.

$$13 = 6(2) + 1 \implies 13 - 6(2) = 1 \implies 2x + 13y = 1, \ with\ x = -6, y = 1.$$

   So we get a solution of $2x + 13y = 31$ by multiplying both sides by 31.

   The solutions are $-6 \times 31 = -186 = x, 1 \times 31 = 31 = y$.

   Check: $(-186)2 + (31)(13) = -372 + 403 = 31$.

   Note that we can only do this because 31 is a multiple of 1.

   If it were not there would be no solutions to $2x + 13y = 31$ in integers because, as we said at the top, 31 has to be divisible by gcd of 2 and 13 because gcd divides any number of form $2x + 13y$.

   Now are there other solutions?

   In fact, if there were one there would be infinitely many.

   Now in the graph of $2x + 13y = 31$ if (x', y') is another point with integer coordinates, then $x' - x, y' - y$ are also integers but $y' - y = m(x' - x)$ means that $y' - y = (-2/13)(x' - x)$ is an integer which basically means $x' - x$ is a multiple of 13. In other words, $x' - x = 13k$ and $y' - y = (-2/13)(13k) = -2k$.

   So for any integer k, if we add $(13k, -2k)$ to $(-186, 31)$ that would be a solution.

Check: $k = 2 \implies (x', y') = (-186, 31) + (13(2), -2(2)) \implies x' = -160, y' = 27$ and $(-160)2 + (27)(13) = -320 + 351 = 31$.

6. A grocer orders apples and bananas at a total cost of \$8.4. If the apples cost 25 cents each and the bananas 5 cents each, how many of each type of fruit did he order.

   Solution: This can be done just like the previous problem using the equation $25x+5y = 840$. First you need to check that gcd of 25 and 5 divides 840.

7. Show, by giving a proof by contradiction, that if four teams play seven games, some pair of teams plays at least two times.

   Solution: Let the four teams be A,B,C,D. Proof by contradiction: Assume each pair plays only once. Then A would play with B,C,D accounting for 3 games, B would play with C and D accounting for two, and then C and D would play one game, for a total of six games. Since they play seven games, we get a contradiction. Thus some pair must play two games.

   This is another application of the Pigeonhole principle.

   Note that you can also get $6 = 4C2 = \binom{4}{2} = (4 \times 3)/2$ using the formula for number of ways to select sets of two things from four (combinations).

8. Use proof by cases to prove that $|xy| = |x||y|$ for all real numbers x and y.

   Solution: There are four cases: $x \geq 0, y \geq 0$; $x < 0, y < 0$; $x \geq 0, y < 0$; $x < 0, y \geq 0$. Show that in each case the identity holds. For example, if $x \geq 0, y < 0$ then $|x| = x, |y| = -y$ and $|xy| = -xy$ and $|xy| = |x||y|$.

9. Prove that $2m^2 + 4n^2 - 1 = 2(m + n)$ has no solution in positive integers.

   Solution: Can prove it with a combination of direct proof and proof by cases.

   First, write this equation as a quadratic equation in the variable n, treating m as a fixed number for the moment. Using quadratic formula,

$$4n^2 - 2n + (2m^2 - 2m - 1) = 0 \implies n = \frac{-(-2) - \sqrt{(-2)^2 - 4(4)(2m^2 - 2m - 1)}}{2(4)}$$

$$\implies (8n - 4)^2 = 4 - 16(2m^2 - 2m - 1) \implies 4 - 16(2m^2 - 2m - 1) \geq 0.$$

   [The RHS has to be ≥ 0 because LHS is a square].

   Now we prove that $4 - 16(2m^2 - 2m - 1) \geq 0$ is not always true, using proof by cases. In fact, we could also use the fact that it needs to be a perfect square of a natural number, but we would use that only for $m = 1$.

   If $m = 1$ then $2m^2 - 2m - 1 = -1$ and $4 - 16(-1) = 20$ is not a perfect square.

   If $m \geq 2$, then clearly $2m^2 - 2m - 1 > 1$ and $4 - 16(2m^2 - 2m - 1) < 0$. (If you want, graph the parabola $y = 2m^2 - 2m - 1$ to convince yourself).

   So in all cases, the equation has no solution in natural numbers.

10. Fill in the details of the following proof that there exist irrational numbers a and b such that a^b is rational:

    Proof Let $x = y = \sqrt{2}$. If x^y is rational, the proof is complete. (Explain.) Otherwise, suppose that x^y is irrational. (Why?) Let $a = x^y$ and $b = \sqrt{2}$. Consider a^b. (How does this complete the proof?) Is this proof constructive or nonconstructive?

    Solution: This is a constructive proof.

    Note that, in this proof *we are only required to produce an example.*

    In general, it is not enough to give just one example.

    The proof is easy to complete as shown, you only have to note that

$$\left((\sqrt{2})^{\sqrt{2}}\right)^{\sqrt{2}} = (\sqrt{2})^{\sqrt{2}\times\sqrt{2}} = \sqrt{2}^2 = 2.$$

11. Use formula for geometric sum $a + ar + ar^2 + ... + ar^n = a(r^n - 1)/(r - 1)$ to show that $1 + r + ... + r^n < \dfrac{1}{1-r}$ for all $n \geq 0$ and $0 < r < 1$.

    (This shows that even the infinite sum $1 + r + r^2 + ...$ would have a finite sum).

    Solution: using the formula, $1 + r + ... + r^n = 1(1 - r^n)/(1 - r) < 1/(1 - r)$ because when $0 < r < 1$, we have $0 < r^n < 1$ and thus $1 - r^n < 1$.

12. Prove using mathematical induction that $n < 3^n$ for all positive integers n.

    Solution: When $n = 1$, we have $1 < 3^1 = 3$.

    Want to show that $n + 1 < 3^{n+1}$ starting from $n < 3^n$.

    Now $n < 3^n \implies n + 1 < 3^n + 1$ but $3^n + 1 < 3^{n+1} = 3(3^n)$ for all $n \geq 1$ (Prove!).

13. Use mathematical induction to prove that

$$\sum_{j=1}^{n}(-1)^{j-1}j^2 = (-1)^{n-1}\frac{n(n+1)}{2}.$$

    Solution: For $n = 1$ we get $(-1)^{1-1}(1^2) = (-1)^{1-1}(1(1+1)/2) \implies 1 = 1$ so it is true.

    Assuming that it is true for n we get

$$1 - 2^2 + 3^3 - .... + (-1)^{n-1}n^2 = (-1)^{n-1}\frac{n(n+1)}{2}. \tag{1}$$

    Now we need to use this to prove for $n + 1$:

$$1 - 2^2 + 3^3 - .... + (-1)^{n-1}n^2 + (-1)^n(n+1)^2 = (-1)^n\frac{(n+1)(n+2)}{2}. \tag{2}$$

    But we see that the statement for n is contained in the LHS.

Plugging in $1 - 2^2 + 3^3 - .... + (-1)^{n-1}n^2 = (-1)^{n-1}\frac{n(n+1)}{2}$ from (1) into (2) in the LHS we get

$$(-1)^{n-1}\frac{n(n+1)}{2}+(-1)^n(n+1)^2 = (-1)^{n-1}(n+1)\left[\frac{n}{2}+(-1)(n+1)\right] = (-1)^{n-1}(n+1)\left[-\frac{n+2}{2}\right]$$

$$= (-1)^{n-1}(-1)\frac{(n+1)(n+2)}{2} = RHS\ of\ (2).$$

Since we showed that the LHS equals the RHS we are done.

14. Use mathematical induction to prove that $2^n < n!$ for $n \geq 4$.

    Solution: We proved this in class.

    First note that this is not true for $n = 1, 2, 3$. (Check!)

    Now for $n = 4$ (which happens to be the first step here) it is true: $2^4 = 16 < 4! = 24$.

    Now we need to prove $2^{n+1} < (n+1)!$ *assuming* $2^n < n!$. In other words, starting with $2^n < n!$.

    Now, multiplying both sides by 2, we get $2^n \times 2 < n! \times 2 \implies 2^{n+1} < 2n!$.

    From this we get $2^{n+1} < (n+1)!$ because $2n! < (n+1)n! = (n+1)!$.

    You can also start with $2^{n+1} < (n+1)!$ and show it is true by reducing it to $2^n < n!$

15. Use mathematical induction to prove that $n^2 < n!$ for $n \geq 4$.

    Solution: Proof is similar to 12. For $n = 4$ we have $4^2 = 16 < 4! = 24$.

    Assuming $n^2 < n!$ true for n we need to prove that $(n+1)^2 < (n+1)!$.

    $$n^2 < n! \implies n^2+2n+1 < n!+2n+1 \implies (n+1)^2 < n!+2n+1 \implies (n+1)^2 < n!+3n$$

    $$\implies (n+1^2 < n! + (n!)n = n!(n+1) = (n+1)! = RHS.$$

    Please prove by yourself that each step works. For instance, if $n \geq 4$, you can show that $2n+1 < 3n$ and $3n < (n!)n$.

16. Prove by induction that the number of subsets of a set X with N elements is 2^n. The set of all subsets is called the power sets, and is denoted as $P(X)$. So this can be written as $|P(x)| = 2^n$. Note that the empty set and the set X itself are subsets of X.

    Solution:

    If the set has only one element, i.e, $n = 1$, then there are two subsets: the empty set and itself. So $|P(X)| = 2^1 = 2$.

    (Actually we need to start with empty set. In this case there is only one subset, and $|P(X)| = 2^0 = 1$).

    Suppose it is true for n.

If X has $n+1$ elements, fix the $n+1-$th element, say x and divide $P(X)$ into two kinds of subsets: *All the subsets not containing* x are basically the subsets of a set of n elements, namely the first n. So there are 2^n such subsets, based on the assumption for n. *All subsets containing* x can be obtained by adding x to one of the subsets in the previous collection, and each one in the previous collection gives rise to exactly one subset in this new collection. In other words, they are in one to one correspndence and hence there are 2^n subsets in the new collection also. But every subset either contains x or doesn't. So total number of subsets is obtained by adding number of subsets in each collection. So $P(X) = 2^n + 2^n = 2(2^n) = 2^{n+1}$.

17. (Hard Problem) This problem shows that the sum of the *harmonic* sequence 1, 1/2,1/3,... namely $1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n} + ...$ goes to infinity.

    (Thus, unlike the geometric sequence mentioned above, even though the terms get smaller and smaller, their sum goes to infinity).

    Prove by induction that the sum of the first 2^n terms of the harmonic sequence is at least $1 + (n/2)$. That is,

$$H_{2^n} = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{2^n - 1} + \frac{1}{2^n} \geq 1 + \frac{n}{2}.$$

    I will leave this as a challenge problem for those interested.