1. Find n such that $35n = 1 \ mod \ 23$ by using the fact that $(23, 35) = 1$ and that you can express 1 as a linear combination of 23 and 35.

2. (a) Write 300 in base 16 and base 2.

   (b) Write 111010110111 in the decimal system and hexadecimal system (base 16)

3. Write an algorithm that finds the gcd of two natural numbers $m = p^a q^b r^c$ and $n = p^d q^e r^f$ where p, q, r are prime numbers.

4. Find the GCD and LCM of 1001 and 2037.

5. Determine using the "square root theorem" or otherwise whether 229 and 837 are prime or composite. Recall that if a, b, z are positive integers, then $ab \ mod \ z = (a \ mod \ z)(b \ mod \ z) \ mod \ z$. Use this to evaluate $41^{22} \ mod \ 25$.

6. The following procedure is used to check whether a number is divisible by 3. Show why it works: If the sum of the digits mod 3 is zero, then it is divisible by 3. If not, it is not divisible by 3. Example: 113 is not divisible by 3 because 1+1+3 = 2 mod 3. [hint: Write the number in powers of 10 and reduce mod 3]

7. How many ways are there for an airline to assign 10 pilots and 10 stewardesses to 10 flights, if a pilot and a stewardess are assigned to each flight?

   (a) What is the probability that, in the above, among the pilots and stewardesses two of the pilots (say pilot number 1 and 2) are not assigned to two of the stewardesses (say stewardess 1 and 2)?

   (b) How many ways are there to pick a pair consisting of a pilot and a co-pilot from 10 airpersons?

   (c) What is the probability that, in the above, two of the airpersons, say A1 and A2, are selected as either pilot or co-pilot?

   Solution:

   For the main problem the answer is just $(10!)^2$ because it is about arranging the 10 pilots among 10 flights and then 10 stewardesses among 10 flights and multiply them together because the two are independent.

   (a) For this part it is easier to do it without taking complements. There are $P(10, 2) = 90$ ways to arrange Pilots 1 and 2 in two of the flights. Then in each case there are $P(8, 2) = 56$ ways to arrange the stewardesses other than S1 and S2 in those two flights. Then the remaining 8 pilots and 8 stewardesses can be arranged in $(8!)^2$ ways. So totally we have $90 \times 56 \times (8!)^2 = 5040 \times (8!)^2$ ways to do it.

   Using complements: The complement of P1 or P2 not assigned to either S1 or S2 would be $\{(P1, S1), (P2, S2), (P1, S2), (P2, S1)\}$. In other words, $P1$ *or* $P2$ are assigned to one of $S1, S2$ in one of the flights. Each of these can be done in 10 ways – just select one of the 10 flights. The remaining 9 flights can be filled with the remaining 9 pilots and 9 stewardesses in $(9!)^2$ ways. So totally $4(10)(9!)^2 = 40(9!)^2$ ways.

But among these there will be overlaps, namely those containing $(P1, S1)$ *and* $(P2, S2)$ or $(P1, S2)$ *and*$(P2, S1)$. These can be found in $2 \times P(10, 2) \times (8!)^2 = 180 \times (8!)^2$ arrangements. So the complement contains $40(9!)^2 - 180(8!)^2 = (3240 - 180)(8!)^2 = (3060)(8!)^2$ arrangements. So the final answer would be $(10!)^2 - (3060(8!)^2) = (8!)^2[(90)^2 - 3060] = (8!)^2(8100 - 3060) = 5040(8!)^2$ ways. So we get the same answer as what we got without taking complements.

(b) There are $P(10, 2) = 90$ ways to choose a pilot and a copilot. Order matters here because each has a position.

(c) There are 2 ways to select A1 or A2 as pilot and copilot: (A1, A2) or (A2, A1). So the probability of that happening is $2/90 = 1/45$.

8. A restaurant menu comprises six choices of appetizers, eight choices of entree and four choices of beverages. How many selections of pairs of appetizer + beverage can one have out of the menu? Do the same for appetizer+entree and entree+beverage.

9. In the senate there are senators from 4 parties, Red, Blue, Green and White. Each party has 25 senators.

   (a) If a random team of five basketball players is picked from this senate, what is the probability that the two forwards will be from the same party and the three other players are also from the same party?

   Solution:

   For the sake of this problem, you can assume that the positions are different so we not only need to pick five people but also arrange them in the five positions. So we have to use permutations.

   Number of ways to pick five from the 100 senators is just $P(100, 5)$. Here we don't need to worry about the parties.

   Now the forwards can be all from one party and the other three all from one party that is different from the party of the forwards. This can be done in $C(4, 2) \times P(25, 2) \times P(25, 3)$ ways because the two parties can be picked in $C(4, 2)$ ways from among the 4.

   But the problem doesn't specify that the other three are from a different party as the forwards! So the same can happen in another case, where the two parties are the same. If all five are from same party it can be done in $4 \times P(25, 5)$ ways because the five can be from one of four parties.

   So finally the probability of two from one party and other three from same party is given by

$$\frac{(C(4, 2) \times P(25, 2) \times P(25, 3)) + (4 \times C(25, 5))}{P(100, 5)}$$

10. In how many ways can 10 identical math books be distributed between 3 students? What if each gets atleast 2 books?

Solution: This is just a partition problem. It is about how to divide up the number 10 into three parts, so we need two partitions. The answer is $C(10+2, 2) = C(12, 2) = (12 \times 11)/2 = 66$.

If each gets atleast 2 books then there are 4 more books left to be divided among the three, so that can be done in $C(4 + 2, 2) = C(6, 2) = 15$ ways.