Fall 06 Test 4 Solutions

1. (20 points) A relation R from a set $\{a, b, c\}$ to itself is given by

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Check whether R is an equivalence relation.

Soln: Call the matrix A.It is easy to see that R is reflexive because A has 1's in the diagonal. It is also easy to see that it is symmetric. To check whether it is transitive, we need to check if A_{ij} is non-zero whenever A^2_{ij} is non-zero. Now $A^2 = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ 0 & 2 & 2 \end{pmatrix}$. Then $A^2_{12} = 1$ but $A_{12} = 0$.

2. (20 points) Write an algorithm that takes a natural number $n > 1$ as input and outputs the first prime number that divides n. You may assume there is a function $isprime(k)$ that returns True if k is prime and False otherwise, as well as another function $mod(n, k)$ that returns the remainder when n is divided by k. [For example, $mod(10, 3) = 1$]. Soln:

firstprimedivisor(n) { k=2
  while $((k \leq n) \wedge (isprime(k) = True))$
    if (mod(n,k)=0)
    return k
  k=k+1
}

3. (20 points) Write an algorithm that checks whether an $n \times n$ matrix represents a reflexive relation from $\{1, 2, 3, \ldots, n\}$ to $\{1, 2, 3, \ldots, n\}$. Assume the matrix has only 1's and 0's.

Soln:
reflexive(A) {
  for k=1 to n {
    if $(A_{kk} = 0)$
    return "False"
  }
  return "True"
}

4. (20 points) Trace the following algorithm with $n = 8$:
Input: n
Output: walk(n)
walk(n) {
  if $(n == 1 \vee n == 2)$
    return n
  return walk(n-1)+walk(n-2)
}

Soln:
Cycle 1: walk(7)+walk(6)
Cycle 2: (walk(6)+walk(5)) + (walk(5)+walk(4))
[walk (7) in the output from cycle 1 returns walk(6) +walk(5) and so on]
Cycle 3: ((walk(5)+walk(4))+(walk(4)+walk(3)))
+((walk(4)+walk(3)) +(walk(3)+walk(2)))
= walk(5)+3walk(4)+3walk(3) + 2
Cycle 4: (walk(4)+walk(3))+(3walk(3)+3walk(2))+
(3walk(2)+3walk(1))+2
= walk(4) + 4walk(3)+6(2)+5 = walk(4)+4walk(3)+17
Cycle 5: walk(3)+walk(2)+4walk(2)+4walk(1)+17
= walk(2) + walk(1)+ 5(2) +21 = 2+1+10+21 = 34.
Check that 34 is indeed the right term.

5. (20 points) Prove using induction that the algorithm of problem 4 is correct and calculates the numbers in the sequence 1,2,3,5,8,13,... defined by $s_n = s_{n-1} + s_{n-2}$ for $n \geq 2$. [The Fibonacci sequence].

Soln: For $n = 1, 2$ we get $s_1 = 1, s_2 = 2$. Suppose $s_1, s_2, \ldots, s_{n-1}$ have all been calculated [Note that we are using strong induction here]. Then walk(n-1)+walk(n-2) can be calculated.