Fall 2017 Final Exam Solutions Discrete Structures 181-01 Sitaraman 12/8/17

1. Given $U = \{1, 2, 3, ..., 10\}, A = \{1, 4, 7, 10\}, B = \{1, 2, 3, 4, 5\}$, and $C = \{2, 4, 6, 8\}$ list the elements of each set: (5 points each)

   (a) $\overline{B} \cap \overline{(C - A)}$ (b) $A \cap (BUC)$ (c) $A \times (B \cap C)$.

   (d) $\mathcal{P}(B \cap C)$ (power set, i.e, set of all subsets of $B \cap C$)

   Solution:

   a) $\overline{B} \cap \overline{(C - A)} = \overline{B \cup (C - A)}$ using DeMorgan's law.

   $\overline{B \cup (C - A)} = \overline{\{1, 2, 3, 4, 5\} \cup (\{2, 4, 6, 8\} - \{1, 4, 7, 10\})} = \overline{\{1, 2, 3, 4, 5\} \cup (\{2, 6, 8\})}$
   $= \overline{\{1, 2, 3, 4, 5, 6, 8\})} = \{7, 9, 10\}$.

   You can do this without DeMorgan as well.

   b) $A \cap (BUC) = \{1, 4, 7, 10\} \cap \{1, 2, 3, 4, 5, 6, 8\} = \{1, 4\}$.

   c) $A \times (B \cap C) = \{1, 4, 7, 10\} \times \{2, 4\} = \{(1, 2), (1, 4), (4, 2), (4, 4), (7, 2), (7, 4), (10, 2), (10, 4)\}$.

   d) $\mathcal{P}(B \cap C) = \mathcal{P}(\{2, 4\}) = \{\{2, 4\}, \{2\}, \{4\}, \{\ \}\}$.

2. (10 points) Use the logical equivalence of $\neg(p \to q)$and $p \wedge \neg q$ to write the negation of "If Jerry receives a scholarship, then he goes to college," symbolically and in words.

   Solution: (Example 1.3.14)

   We let p denote "Jerry receives a scholarship," and q denote "Jerry goes to college." The given proposition can be written symbolically as $p \to q$. Its negation is logically equivalent to $p \wedge \neg q$. In words, this last expression is "Jerry receives a scholarship and he does not go to college."

3. Determine the truth value of following: (10 points each)

   (a) $\forall x \forall y((x < y) \to (x^2 < y^2))$. The domain of discourse is $\mathbb{R} \times \mathbb{R}$.

   (b) $\forall x(x < x^2)$. The domain of discourse is the set of positive real numbers.

   Solution:

   (a) This is not true. Proof is using counterexample: $-3 < -2$ but $(-3)^2 > (-2)^2$. In fact this will not be true for any two negative numbers and will be true for any two positive real numbers.

   (b) This is not true. Proof is using counterexample: $0.5 > 0.5^2$. In fact this will not be true for any $x \in (0, 1)$ and will be true for any $x \in (1, \infty)$.

4. (10 points) Show, by giving a proof by contradiction, that if four teams play seven games, some pair of teams plays at least two times.

   Solution:

   There are 6 pairs among four teams. Since they play seven games, at least one pair must play two times. If not, they only play six games.

5. (10 points) Show by giving a proof by cases, that for every integer $n, n^3 + n$ is even.

   Solution:

   If n is even then both n^3 and n are even and a sum of two even numbes is even so $n^3 + n$ is even.

   In n is odd then it is of form $2m + 1$ for some integer m and $n^3 = (2m + 1)^3 = 8m^3 + 6m + 12m^2 + 1 = 2(4m^3 + 3m + 6m^2) + 1$ which is thus of the form $2k + 1$ and hence odd. So then $n^3 + n = 2m + 1 + 2k + 1 = 2(m + k) + 2 = 2(m + k + 1)$ and thus it is even as well.

6. (20 points) For the hash function $h(x) = x \mod 13$, show how the data 784, 281, 1141, 18, 1, 329, 620, 43, 31, 684 would be inserted in the order given in initially empty cells indexed 0 to 12.

   Solution: (From chapter 3 self-test) You look at the number mod 13 and assign it to that numbered cell or next available cell. For example 784 is 4 mod 13 and so it is assigned to cell number 4. (a :b means store item a in cell b.) $1 : 1$, $784 : 4$, $18 : 5$, $329 : 6$, $43 : 7$, $281 : 8$, $620 : 9$, $1141 : 10$, $31 : 11$, $684 : 12$

7. For the sequence b defined by $b_n = n(-1)^n, n \geq 1$, find:

   (a) (4 points) $\sum_{i=1}^{4} b_i$.

   (b) (10 points) Formulae for the sequence c defined by $c_n = \sum_{i=1}^{n} b_i$ when n is odd and n is even (separately). Your answer should be a function of n.

   (c) (6 points, a bit harder) A single formula for c_n that works for both even and odd n.

   Solution:

   a) $\sum_{i=1}^{10} b_i = 1(-1)^1 + 2(-1)^2 + 3(-1)^3 + 4(-1)^4 = -1 + 2 - 3 + 4 = 2.$

   b) You can see that the sum will be $n/2$ for even n. This is because you can pair successive terms to get (-1+2)+(-3+4)+... = 1+1+...+1 and there will be $n/2$ pairs giving $n/2$ times 1.

   Now for odd n, we will have $c_n = \frac{n-1}{2} + n(-1)^n =$ because all the terms upto $(n-1)(-1)^{n-1}$ will add upto $(n-1)/2$ because $n - 1$ is even. But if n is odd then $(-1)^n = -1$ so we get $c_n = \frac{n-1}{2} - n = \frac{-n-1}{2}$.

   (c) We can write a formula that works for both even and odd numbers as follows: $c_n = (-1)^n \frac{n}{2} + (1/2)\frac{(-1 + (-1)^n)}{2}$.

8. Let R be an equivalence relation on a set A. Define a function f from A to the set of equivalence classes of A by the rule $f(x) = [x]$.

   (a) (6 points) Show that this is an onto function.

   (b) (8 points) Given an example of a set A with an equivalence relation R such that f is a one-one function.

   (c) (6 points) When do we have $f(x) = f(y)$?

   Solution:

   a) This is onto because every element x belongs to an equivalence class, namely $[x]$ because R has to be reflexive and xRx has to be true.

   b) Let $A = \mathbb{R}$, be the set of real numbers. Then define R by $xRy \iff x = y$. Then each element is its own equivalence class, i.e, $[x] = \{x\}$. In this case clearly f is one-one.

   c) $f(x) = f(y) \iff [x] = [y] \iff x \in [y]\ or\ y \in [x] \iff xRy.$

9. (20 points) Write a pseudocode for the Euclidean algorithm that takes any two non-negative integers (not both zero) as input and outputs their gcd.

   Solution: (Algorithm 5.3.3)

   This algorithm finds the greatest common divisor of the nonnegative integers a and b, where not both a and b are zero.

   Input: a and b (nonnegative integers, not both zero)

   Output: Greatest common divisor of a and b

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gcd(a, b) {
( make a largest)
if (a<b)
swap(a, b)
while (b ≠ 0) {
r = a mod b
a = b
b = r
}
return a
}
```

10. Using a deck of 52 cards, four bridge hands each with 13 cards are dealt. For the following, calculate only as much as you can. As long as your formula is right, you will get full credit.

(a) (5 points) Find total number of bridge deals possible. In other words, how many 13 card hands are possible?

(b) (5 points) How many bridge hands with all four aces?

(c) (10 points) Find the probability of obtaining a bridge hand with $6-5-2-0$ distribution, that is, six cards in one suit, five cards in another suit, two cards in another suit, and no cards in the fourth suit.

Solution:

(a) $C(52, 13)$.

(b) $C(48, 9)$. Select the four aces in one way and then remaining 9 cards in $C(48, 9)$ ways.

(c) (Chapter 6 self-test) Choose suit with six cards in four ways and then the six cards in that suit, then choose suit with five cards in three ways and then choose the five cards in that suit, and finally choose suit with two cards in two ways and then choose the two cards in that suit. Finally divide by total number of hands possible – your answer from (a).

$$\frac{4C(13,6)\times 3C(13,5)\times 2C(13,2)}{C(52,13)}$$

11. (10 points) In how many ways can 10 distinct books be divided among three students if the first student gets five books, the second three books, and the third two books?

Solution: (Problem 6.3.10) If the students are labeled A, B, C then this is really number of ways to arrange AAAAABBBCC and that is

$$\frac{10!}{5!3!2!} = \frac{10.9.8.7.6}{6.2} = 5.9.8.7 = 45(56) = 2520.$$

You can also say that this is number of ways to choose the books for the first student times number of ways to do that for second and so on:

$$C(10, 5) \times C(5, 3) \times C(2, 2).$$

12. (20 points) Prove by induction that $n! < \dfrac{n^n}{2^n}$. Note that this is not true for $n = 1, 2, 3, 4, 5$ but true for all $n \geq 6$. You only need regular induction, though.

Solution:

True for $n = 6 : 6! = 720 < 6^6/2^6 = 3^6 = 27^2 = 729$.

Let $n \geq 6$ and assume $n! < n^n/2^n$.

Need to prove $(n+1)! < \dfrac{(n+1)^{n+1}}{2^{n+1}}$.

We have by induction hypothesis (multiply both sides of $n! < n^n/2^n$ by $n+1$)

$$(n+1)! = n!(n+1) < \frac{(n^n)(n+1)}{2^n} \quad \text{(eqn A)}.$$

We can actually show that

$$\frac{(n^n)(n+1)}{2^n} < \frac{(n+1)^{n+1}}{2^{n+1}} \text{ and this along with eqn A will complete proof.}$$

After factoring out $(n+1)/2^n$ we get

$$n^n < \frac{(n+1)^n}{2} \implies \frac{(n+1)^n}{n^n} > 2 \implies \left(1+\frac{1}{n}\right)^n > 2 \implies 1 + n(1/n) + ... > 2.$$

In the last implication we have, after expanding using binomial formula or otherwise, $1 + 1 + (\textit{positive quantity}) > 2$, which works as long as $n > 2$.

FYI:

Note that $\left(1+\dfrac{1}{n}\right)^n \to e$ as $n \to \infty$.

Since this is an increasing sequence (show!) we can say that $\left(1+\dfrac{1}{n}\right)^n < e$ for all n.

Thus using same kind of argument as above we can show that $n! > n^n/e^n$.

$$\text{So we have } \frac{n^n}{e^n} < n! < \frac{n^n}{2^n}.$$