Methods of Proof

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Class Notes

What is a *Proof* ?

- A series of logical statements together called an ***argument*** that results in establishing the truth of a mathematical statement of fact called **lemma** or **theorem**.
- The statements should each be true and one statement should lead to the next statement.
- They should use only the **definitions** and **axioms** that have been previously stated and the **hypotheses** that have been made in the lemma or theorem to be proved.

Proofs are not always easy!

- Sometimes proofs can take years, as dramatized in the video clip from "Good Will Hunting"
- The statement

" $x^n + y^n = z^n$ has no solution if

x, y, z are integers and *n is an integer > 2* "

was made in 1637 by Fermat but proved by Wiles only in 1994!

- Example: $3^2+4^2 = 5^2$ and many other ***Pythagorean triples*** *but* $x^3 + y^3 = z^3$ is *never true for integers!*

Direct proof

Example(2.1.7)

If m and n are even, then m+n is even

m even $\Rightarrow m = 2m_1$ for some $m_1 \in \mathbb{Z}$

n even $\Rightarrow n = 2n_1$ for some $n_1 \in \mathbb{Z}$

So then $m+n = 2(m_1+n_1)$ is also even.

In this proof we have only used the definition of an even integer

Example:(Proof by Contradiction) Euclid's proof on prime numbers

Statement: (3^{rd} cy BC) ***There are infinitely many prime numbers***

Proof: (using the method ***Proof by Contradiction***)

1. Suppose there are only finitely many, say p_1, p_2, p_3,…., p_n are ALL the n natural numbers that are divisible by 1 and themselves. Let P be their product.
2. Then P+1 leaves a remainder of 1 when you divide by any of the primes, so it is divisible exactly only by 1 and itself!
3. Therefore P+1 is a prime number that is not equal to any of the other prime numbers.
4. Therefore there are more than n prime numbers.

The contradiction means the statement should be true.

What are the **definitions and axioms** that we used?

Proof by contrapositive

Idea: To prove that p → q, we show that ¬ q → ¬ p

Example: (2.2.5)

$$x + y + z \geq 3 \rightarrow x \geq 1 \vee y \geq 1 \vee z \geq 1$$

Try to prove it!

Proof of Problem 2.2.5

Contrapositive of

$x \geq 1 \vee y \geq 1 \vee z \geq 1$

is $x < 1 \wedge y < 1 \wedge z < 1$

In that case

$x + y + z < 3$

$= \neg (x + y + z \geq 3)$