8/25/2017
Class Notes

Review of what we did wednesday

- Set theory and logic together were used to provide a rigorous foundation for math
- Logicians also were able to find the limitations of logic and mathematical structures
- Removing parallel lines axiom from geometry and replacing it with a suitable axiom results in non-Euclidean geometry. Shows that a mathematical system depends on its axioms.
- Godel's incompleteness theorem showed further limitations (but not same as dependence of system on axiom)
- But once you know the limitations, you have a very good foundation

Review (page 2)

- With such a firm foundation, mathematics can be made almost mechanical
- In future computers could verify theorems and even come up with new ones
- Computers are a natural next step in this evolution of math. They are basically machines that can do math, and they also depend on math to do everything.

Review (page 3)
Basic logical statements

- We saw how to write logical statements symbolically using NOT, AND, and OR operators and vice versa.
- We proved De Morgan's laws for logic
- NOT(p OR q) ≡ (NOT p) AND (NOT q)
- Where ≡ means "is equivalent to"

Other symbols

$\neg \quad \lor \quad \land \quad \forall \quad \cup \quad \cap \quad \exists \quad \nexists \quad \rightarrow \quad \leftrightarrow$

De Morgan's Law for sets

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

Complement of union equals intersection of complements

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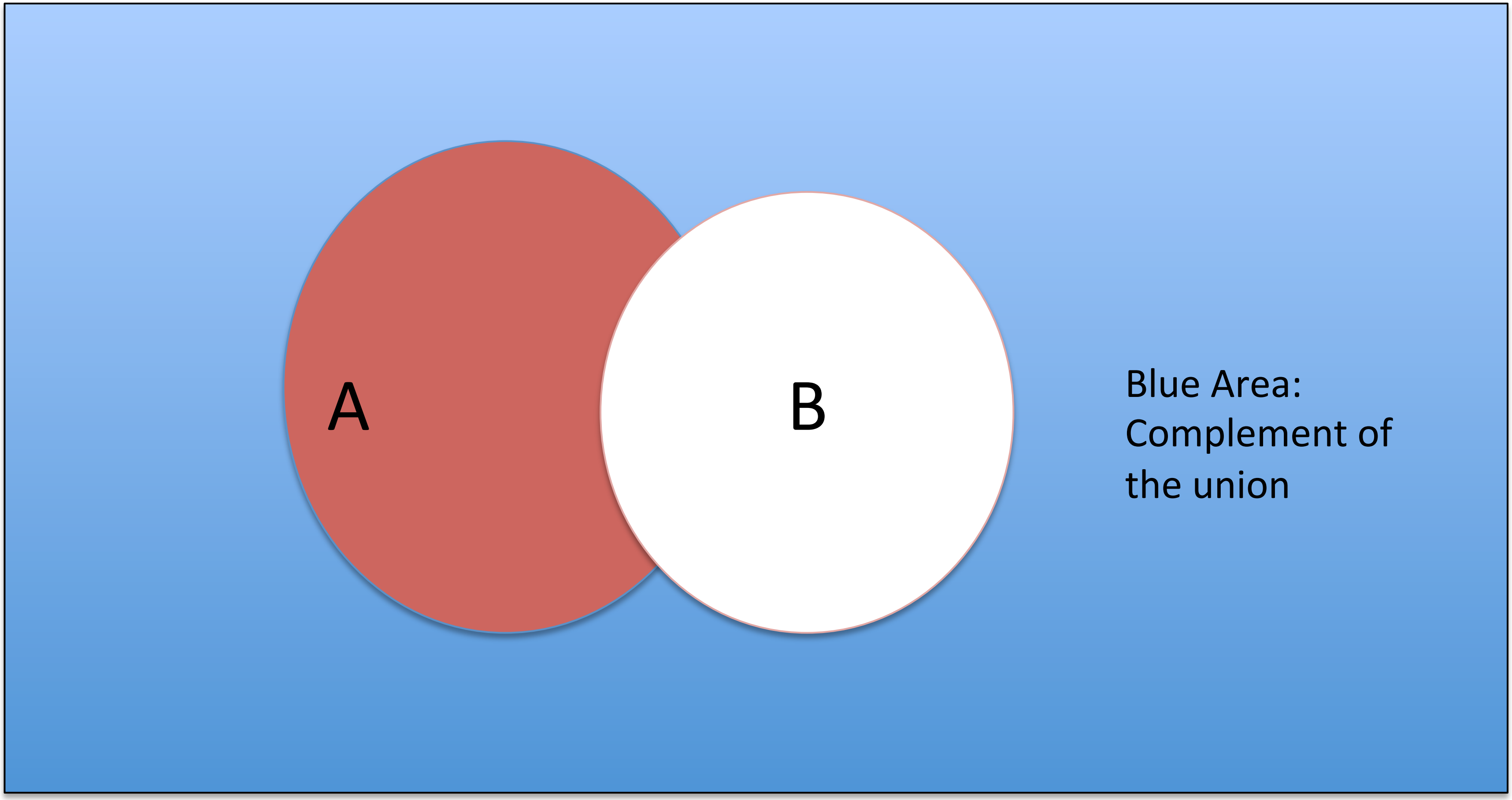
De Morgan’s Law in pictures



Conditional statements (page 1)

- Let m,n,p,a,b all be natural numbers.
- Let P be the statement

  "n is a prime number"
- Let Q be the statement

  "$n = a^2+b^2$ for some a and b"
- Let R be the statement

  " n is of the form 4m+1"

Conditional statements (page 2)

Fermat's Theorem on sums of two squares

A prime number is a sum of two squares

if and only if

that prime number is of the form 4m+1

Conditional statements (page 3)

Fermat's Theorem in Symbols

$$(P \wedge Q) \leftrightarrow (P \wedge R)$$

Conditional statements (page 4)

Based on Fermat's theorem, which of the following are true?

A. Every natural number that is a sum of two squares is a prime number
B. Every natural number of the form 4m+3 is not a sum of two squares
C. Every prime number of the form 4m+3 is not a sum of two squares
D. Every natural number of form 4m+1 is a sum of two squares.

Answers to questions from previous page

C is true and it is the contrapositive of the statement “P AND Q -> P AND R.” More on that in an ensuing slide. A, B and D cannot be answered based only on Fermat’s theorem’s statement. The reason I put them there was twofold:

1. To show the scope of the statement and to show how to understand the scope of a statement.
2. To show some interesting facts from theory of numbers

So are A, B and D true or not? (Just to pique your curiosity)

- Here is what is true (remember, this is outside the scope of the statement of Fermat's theorem, which is concerned with prime numbers):
- 25 = $4^2+5^{2,}$ so that is a counter-example for A.
- 9 is not the sum of two squares, so that gives a counterexample for D. (0 is not a natural number).
- It is true that if n is of form 4m+3 then it is not the sum of two squares. Proof is elementary. Try!

Conditional statements (page 4)

CONTRAPOSITIVE

OF A

CONDITIONAL STATEMENT

IF P IMPLIES Q, THEN NOT Q IMPLIES NOT P

Example of contrapositives

Statement:

If sun I shining then it will be bright outside.

Contrapositive:

If it is not bright outside then sun is not shining.

Difference between ≡ and ⟷

$p \equiv q$ means p and q are logically equivalent.

The statements always have the same logical value (T or F) regardless of the values of their components.

$p \leftrightarrow q$ (p iff q) is only concerned with the relationship – whether one implies the other.

Example in next page.

Difference between ≡ and ↔ : Example

- The statements "A implies B" and the statement "not B implies not A" are logically equivalent, regardless of what A and B are or whether A and B are true.
- But it would be silly to say "A implies B" iff "not B implies not A" even if that is true, because they are really two ways of saying same thing. (continued next page…

Difference between ≡ and ⟷ : Example (cont.d from previous page)

On the other hand the two statements "P : The sun is shining" and "Q: It is daytime" are related by iff.

P⟷Q because if sun is shining it is daytime and if it is daytime the sun must be shining. But we cannot say P ≡ Q. The two are not logically equivalent.

Being daytime is related to the sun shining but it is not just another way to say that the sun is shining.