1/13/2018 Spring 2018, Calculus I Brief answers to Final Exam Sitaraman

1. There are only two sides that need to be fenced. The hypotenuse is covered by the river. Let x be the length of one of them. Then since total length of fence is 1000, the other side is $1000 - x$.

   The Area $A(x)$ is therefore $A(x) = lw/2 = x(1000 - x)/2$

   $= (1000x - x^2)/2 = 500x - (x^2/2)$.

   To find the maximum area we find the derivative.

   We have $A'(x) = 500 - x$. Derivative is zero when $x = 500$.

   Since the function is well defined everywhere $x = 500$ is the only critical point.

   The maximum would then happen either at the critical point or the boundary points. The length has to be between 0 and 1000 so those are the boundary values of x. When you plug in 0 or 1000 into $A(x)$ you get 0, so $A(500) = 500(1000-500)/2 = 250000/2 = 125000$ is the maximum area.

   The dimensions that produce this area are 500 and 500 for both sides.

2. For (a) you would add the values at 1,1.25,1.5 and 1.75 because you are starting at the left endpoints and length of each interval is (2-1)/4 = 0.25. So you start at 1 and add .25 each time. Answer would be $f(1)+f(1.25)+f(1.5)+f(1.75) = \frac{1}{1} + \frac{1}{1.25} + \frac{1}{1.5} + \frac{1}{1.75}$.

   This would be an overestimate because, as you can see from the graph, the area under the graph is *inside* the rectangles.

   If, on the other hand, you were using the right end points, then it would be an underestimate because in that case the rectangles would be inside the area under the graph.

3. Parts (a) and (b) are just straight applications of chain rule.

   $$(e^x\sqrt{x})' = e^x\sqrt{x} + e^x(1/(2\sqrt{x})) \ ; \ \left(\frac{\tan x + \sec x}{\cos x - \sin x}\right)' =$$

   $$= \frac{(\sec^2 x + \sec x \tan x)(\cos x - \sin x) - (\tan x + \sec x)(-\sin x - \cos x)}{(\cos x - \sin x)^2}.$$

   When simplifying please make sure to subtract all the terms within parantheses.

   Part (c) : Use chain rule:

   $$(\ln(\ln x))' = \frac{1}{\ln x}(1/x) = \frac{1}{x \ln x}$$

   Part (d) Use the limit formula definition of $f'(-1)$ and plug in $x = -1$:

   $$f'(-1) = \lim_{h\to 0} \frac{f(-1+h) - f(-1)}{h} = \lim_{h\to 0} \frac{3(-1)^2 h + 5h^2}{h} = \lim_{h\to 0} 3 + 5h = 3.$$

4. Upon differentiating all with respect to x we get $2x - xy' - y - 2yy' = 0 \implies 2x - y = xy' + 2yy' \implies y' = (2x - y)/(x + 2y)$.

   Plug in $x = 2, y = 1$ to get slope at (2,1) and then use point slope formula to get equation of tangent line at that point.

5. To find critical numbers, see where derivative is zero or undefined.

$$f'(x) = (6\sqrt[3]{x^2}-2x+4)' = 6(x^{2/3})'-(2x)'+(4)' = 6(2/3)x^{-1/3}-2 = \frac{4}{x^{1/3}}-2 = \frac{2(2-\sqrt[3]{x})}{\sqrt[3]{x}}$$

   This is undefined when $\sqrt[3]{x} = 0 \implies x = 0$.

   It is zero when numerator is zero, which happens when $2 = \sqrt[3]{x} \implies 2^3 = 8 = x$. So 0 and 8 are the critical points.

   To find absolute maximum and minimum in [-8,1] compare the values at the critical points and boundary points: $f(-8), f(0), f(1)$. 8 is not included because it is not in the interval of interest.

6. x^3 satisfies the conditions of the mean value theorem (MVT) in [-1,1] because it is continuous in the closed interval [-1,1] and differentiable in every point of the open interval (-1,1). In fact it is continuous and differentiable everywhere on the real number line.

   To find numbers c that satisfy conclusion of MVT we try to solve for c from the equation $f'(c) = (f(b) - f(a))/(b - a) = (f(1) - f(-1))/(1 - (-1)) = (1^3 - (-1)^3/2 = (1 - (-1))/2 = 2$. But $f'(c) = 3c^2 \implies 3c^2 = 1$ which gives $c = \pm\sqrt{1/3}$. Both $1/\sqrt{3}$ and $-1/\sqrt{3}$ are in [-1,1] and they are the desired values of c.

7.

$$(a) \lim_{x\to-4} \frac{\sqrt{x^2+9}-5}{x+4} = \lim_{x\to-4} \frac{\sqrt{x^2+9}-5}{x+4}\frac{\sqrt{x^2+9}+5}{\sqrt{x^2+9}+5} = \lim_{x\to-4} \frac{x^2+9-5^2}{(x+4)(\sqrt{x^2+9}+5)}$$

$$= \lim_{x\to-4} \frac{x^2-16}{(x+4)(\sqrt{x^2+9}+5)} = \lim_{x\to-4} \frac{x-4}{\sqrt{x^2+9}+5} = -8/10 = -4/5 = -0.8.$$

$$(b) \quad \lim_{x\to0} \left(\frac{1}{x} - \frac{1}{x^2+x}\right) = \lim_{x\to0} \frac{(x^2+x)-x}{x(x^2+x)} = \lim_{x\to0} \frac{x^2}{x(x(x+1))} = \lim_{x\to0} \frac{1}{x+1} = 1.$$

   In all of the above note that we cannot simplify these fractions the way we did if we plugged in $x = 0$. The reason it works is because when we try to find the limit, we are looking at points near 0.

(c) Divide all terms above and below by x^2. As $x \to \infty$ the numerator goes to 73 and the denominator behaves like x itself and hence goes to infinity. But $73/x \to 0$ as $x \to \infty$.

(d) $-1 \leq \sin(\pi/x) \leq 1$ because the sine function is always between -1 and 1. So $\sqrt{x}e^{-1} \leq \sqrt{x}e^{\sin(\pi/x)} \leq \sqrt{x}e^{1}$ and that means $\sqrt{x}e^{\sin(\pi/x)} \to 0$ because the terms on both sides of it go to 0 as $x \to 0^+$.

8. In part (a) you can factorize the numerator as $(x+3)(x-2)$. The denominator $|x-2|$ can be replaced by $x-2$ when $x \to 2^+$ because on the right side of 2 we have $x-2>0$ and $|x-2| = x-2$. When you divide out $x-2$ you get $x+3$ which goes to the limit 5 as $x \to 2^+$.

   Similarly in part (b) $|x-2| = -(x-2)$ when $x \to 2^-$ and the limit is -5.

   Since the limits from teh two sides are different the limit as $x \to 2$ doesn't exist.

9. In order for f to be continuous, the value at 2 must be the same when you approach from left or right. At all other values of x the function is continuous because it is given by polynomials.

   Since $2xA \to 4A$ when $x \leq 2$ and $x^2 + A = 4 + A$ when $x > 2$ we need $4A = 4 + A$ which means $3A = 4$ or $A = 3/4$.

   For (b) again, the derivative exists for all values other than 2 so we only need it to be differentiable at 2. This will happen if $(f(2+h) - f(2))/h$ approaches the same value whether $h \to 0$ from positive or negative side, which is another way of saying we approach 2 from the right or left because $2 + h \to 2$ as $h \to 0$. But the limit from the left is just the derivative of $2xA$ and so it is $2A$. The limit from the right is $(x^2 + A)' = 2x = 4$ when $x = 2$. So $f'(2)$ exists if $2A = 4$ or $A = 2$. But from (a) we need $A = 3/4$ just for the function to be continuous. So it won't be differentiable at 2 and hence not for all x.

10. Find $f'(0)$ and use it to approximate $\sqrt{0.9}$ by using the linear approximation formula (really equation of tangent at $x = 0$) $f(x) \simeq f(0) + f'(0)x$. Note that here $x = -0.1$ so that $1 + x = 1 - 0.1 = 0.9$. You will get that $\sqrt{0.9} = 1 + (1/2)(-0.1) = 0.95$ approximately.

11. Let y be the distance directly from the radar to the rocket (imagaine you have a rope tied from the radar straight to the rocket) and h the height of the rocket. Then $y^2 = 5^2 + h^2$. To find rate of change (speed) we need to differentiate both sides with respect to time. We get $2y(dy/dt) = 2h(dh/dt))$. It is given that $h = 4$ and $dy/dt = 2000mph$.

    Using $y^2 = 5^2 + h^2$ we get $y^2 = 41 \implies y = \sqrt{41}$. Plugging in all the values into $2y(dy/dt) = 2h(dh/dt))$ and solving, we get $dh/dt = 2\sqrt{41}(2000)/(2 \times 4) = 500\sqrt{41}$.

12. True or False questions.

(a) False. $f''(0) = 0$ for $f(x) = x$ but clearly the line $y = x$ does not have an inflection point at 0. In general, inflection point happens where concavity changes direction, and $f''(c) = 0$ is not enough.

(b) False. The absolute value function –with a V shaped graph that has vertex at 0 – is continuous at 0 but not differentiable there. In general the function needs to be smooth. If there is a pointed edge then it won't be differentiable.

(c) False. The limit can be 0 or any fixed number and then graph will have a horizontal asymptote.

(d) False. Derivatives can be equal even if the function differs by a constant number. For example, for any fixed number C both x^2 and $x^2 + C$ will have derivative equal to $2x$. This is why we write the antiderivative as $F(x) + C$.

(e) True. This follows from the very definition of a removable discontinuity.