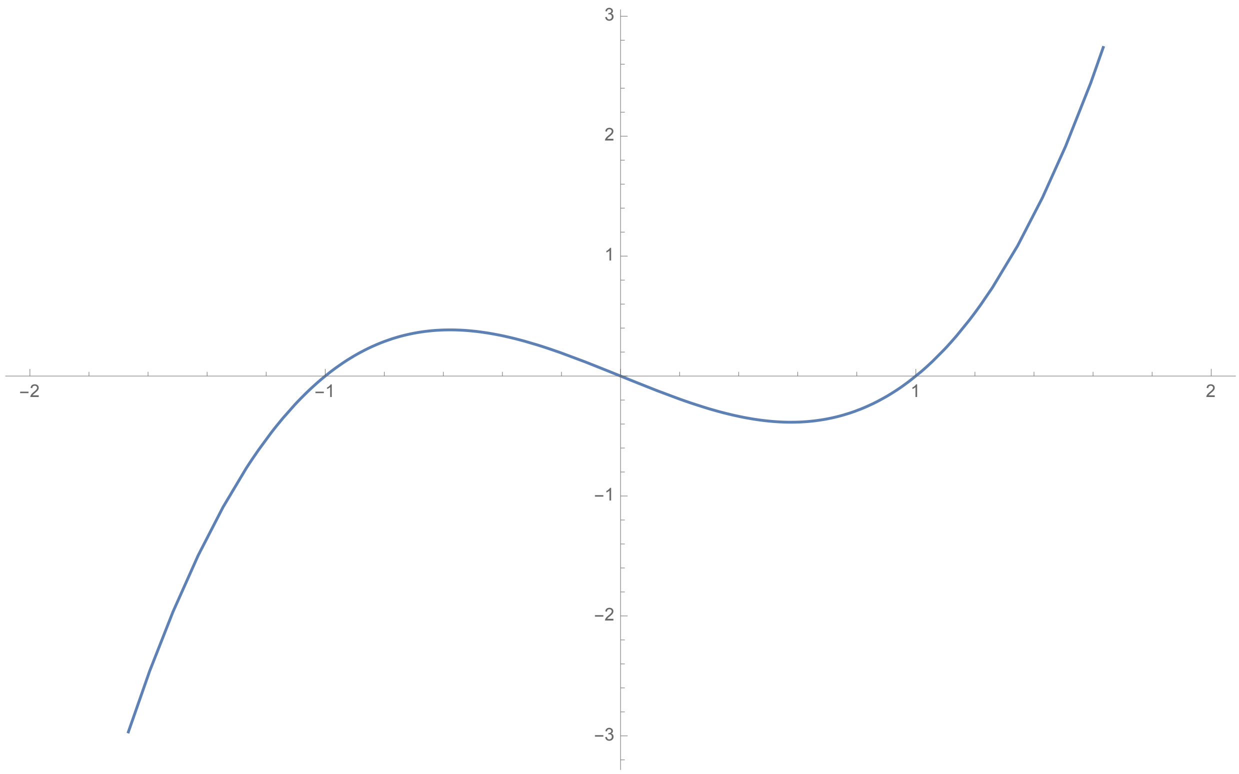
11/30/2017 Fall 2017, Calculus I Test III (Version I) Solutions Sitaraman

Howard University Math Department

Instructions:
PLEASE PROVIDE STEP BY STEP EXPLANATIONS
WRITING ONLY ANSWERS WILL NOT GET FULL CREDIT
Time Limit 50 minutes. Anything you get over 100 points is extra credit.
Please read the questions carefully before answering

1. The graph of a function $f(x)$ is given below. Answer the questions following the graph. You MUST give reasons for your answers. The values can be approximate.



(a) (5 pts) On what intervals is the derivative $f'(x)$ positive and on what intervals is it negative?

(b) (5 pts) On what intervals is $f'(x)$ increasing? Where is the graph concave downwards?

(c) (5 pts) On what intervals is the second derivative $f''(x)$ positive and where is it negative?

(d) (5 pts) List the points where the function has a local maximum, local minimum, or inflexion point.

Solution: This is the graph of $x^3 - x$. (You don't need to say that). It is increasing from $-\infty$ to $-1/\sqrt{3}$ and then from $1/\sqrt{3}$ to ∞. It is decreasing between $-1/\sqrt{3}$ and $1/\sqrt{3}$.

Those two points are its relative maximum and minimum, respectively. The inflexion point is at 0 where it changes from concave down to concave up.

The derivative is increasing after 0 (concave up, second derivative positive) and decreasing before 0 (concave down, second derivative negative).

2. (20 points) The function x^x can be made to be continuous at 0 in the interval $[0, \infty)$ by defining $(0)^0 = Lim_{x \to 0^+} x^x$. What is this limit?

Solution: We use L'Hospital's rule to find this limit. But first we need to convert it to the form 0/0 or ∞/∞.

We do this by taking $\ln y$ where $y = x^x$.

We get $\ln y = x\ln(x) = \frac{\ln(x)}{1/x}$.

This is of the form ∞/∞ as $x \to 0^+$.

Using L'Hospital's rule we get $Lim_{x\to 0^+}\dfrac{\ln(x)}{1/x} = Lim_{x\to 0^+}\dfrac{(\ln(x))'}{(1/x)'} = Lim_{x\to 0^+}\dfrac{1/x}{(-1/x^2)}$

$= Lim_{x\to 0^+} -x = 0$.

So $\ln y \to 0$ and thus $y \to e^0 = 1$.

So the limit should be 1 in order to make the function continuous at 0.

3. (40 points]) The following involve the function $f(x) = 3 + \sqrt{1-t^2}$:

(a) Approximate $\int_0^1 f(t)\ dt$ by dividing [0,1] into five intervals given that $f(0.1) = 3.995, f(0.3) = 3.954, f(0.5) = 3.866, f(0.7) = 3.714, f(0.9) = 3.436$. Your answer will be 3.793 approximately.

(b)Express the integral as an area and then use geometry to find area. Compare with answer from (a).

Solution:

(a) Each of the values given is the value of f at the midpoint of the intervals of length 0.2 given by (0,0.2), (0.2,0.4),...,(0.8,1.0).

The approximation is $f(0.1)(0.2) + f(0.3)(0.2) + ... + f(0.9)(0.2) = 18.965 \times 0.2 = 3.793$ approximately.

(b) Separating the integral into two we get $\int_0^1 f(t)\ dt = \int_0^1 3\ dt + \int_0^1 \sqrt{1-t^2}\ dt$

The first is the area of rectangle between $y = 3$, the $x-$axis, the $y - axis$ and $x = 1$. This is just 3. The second is the area of the quarter of the circle $x^2 + y^2 = 1$ lying in the first quadrant. This is because $\sqrt{1-x^2} = y \implies y^2 + x^2 = 1$. This area is $\pi r^2/4 = \pi/4 = 0.785$. Thus the total area will be 3.785 which is within 0.008 of answer from (a).

4. The power in an external resistor of R ohms that is connected across a battery of E volts with internal resistance r ohms is given by

$$P(R) = \frac{E^2R}{(R+r)^2}.$$

If E and r are fixed but R varies over all positive real numbers,

(a) (4 points) What happens to P as $R \to \infty$?

(b) (8 points) When is P increasing and when is it decreasing? You must use derivatives.

(c)(8 points) Show that $P(R)$ is maximum when the two resistances are equal, namely $R = r$ and that the maximum value of the power is $E^2/4r$. Must use first derivative test.

Solution: To find critical point set $P'(R) = 0$ and solve for R. Remember that E and r are kept constant. We get

$$P'(R) = E^2(R/(R+r)^2)' = E^2\frac{1(R+r)^2 - R(2(R+r))}{(R+r)^4}$$

$$= E^2\frac{R^2 + 2Rr + r^2 - 2R^2 - 2Rr}{(R+r)^4} = E^2\frac{r^2 - R^2}{(R+r)^4}.$$

$P'(R) = 0 \implies r^2 - R^2 = 0$. So we have $r^2 = R^2 \implies r = R$ where we take only positive root because neither r nor R are negative.

We see that when $R < r$ the numerator is positive and since the denominator is always positive the power is increasing to the left of r. Similarly we can see that it is decreasing to the right. So $R = r$ is the local maximum by the first derivative test.

When $R = 0$ the power is also 0 and when $R \to \infty$ the power goes to 0 because the degree of the denominator is smaller and we see that $\frac{R}{(R+r)^2} \to \frac{R/R^2}{(R+r)^2/R^2} \to \frac{1/R}{(1+\frac{r}{R})^2} \to 0$. (divided above and below by highest power, namely R^2). You can also do this using L'Hospital's rule.

We see that at the boundaries $P \to 0$. So $R = r$ is also the absolute maximum and the maximum value of the power when $R = r$ is $E^2r/(r+r)^2 = E^2r/(2r)^2 = E^2/4r$.

5. (extra credit 20 points) Find the area under the curve $y = \frac{2x}{1+x^2}$ between $x = 0$ and $x = 1$. You must use integration by substitution. Your answer will be $\ln 2$.

Solution:

Let $u = 1 + x^2$. Then $du = 2xdx$. Also $u(0) = 1, u(1) = 2$.

$$\text{Area } = \int_0^1 \frac{2x\ dx}{1+x^2} = \int_1^2 \frac{du}{u} = [\ln u]_1^2 = \ln 2 - \ln 1 = \ln 2.$$

We didn't have to write $\ln |u|$ because $1 + x^2$ is always positive.

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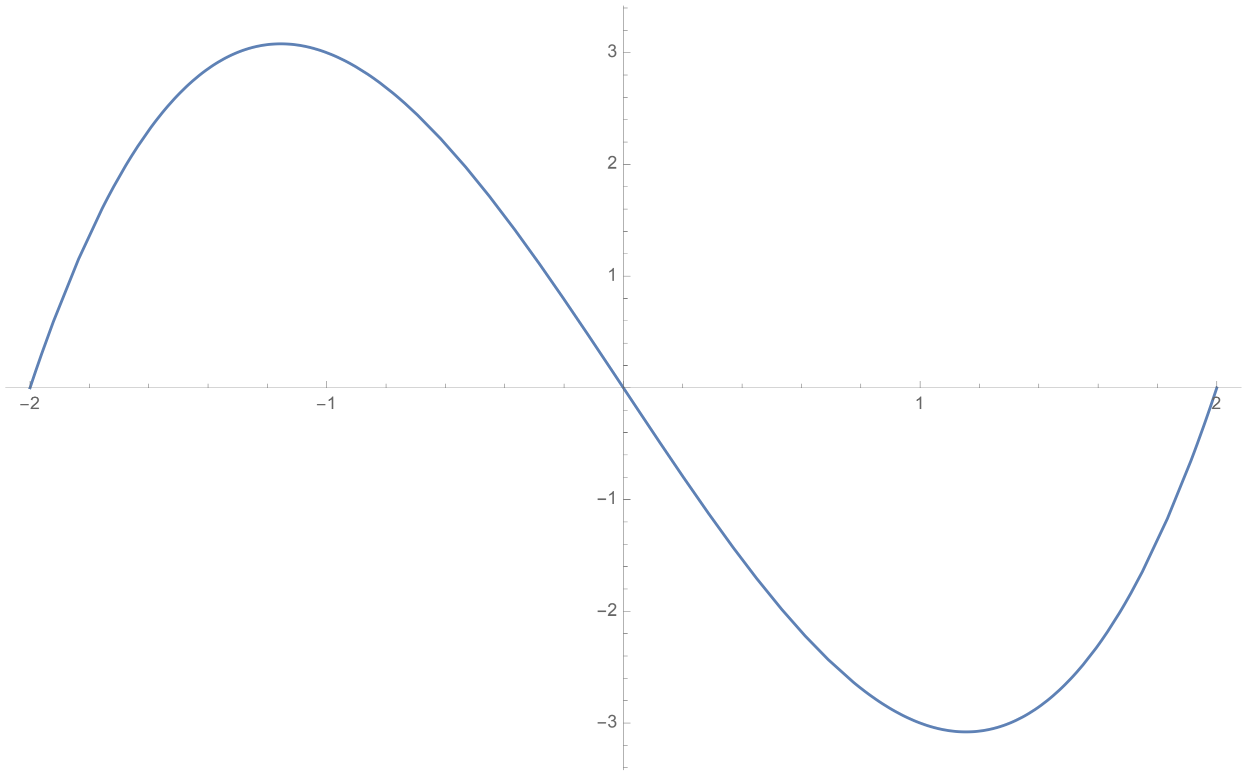

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(d) (5 pts) List the points where the function has a local maximum, local minimum, or inflexion point.

Solution: This is the graph of $x^3 - 4x$. (You don't need to say that). It is increasing from $-\infty$ to $-\sqrt{4/3}$ and then from $\sqrt{4/3}$ to ∞. It is decreasing between $-\sqrt{4/3}$ and $\sqrt{4/3}$.

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The derivative is increasing after 0 (concave up, second derivative positive) and decreasing before 0 (concave down, second derivative negative).

2. (20 points) The function x^{2x} can be made to be continuous at 0 in the interval $[0, \infty)$ by defining $(0)^0 = Lim_{x \to 0^+} x^{2x}$. What is this limit?

Solution: We use L'Hospital's rule to find this limit. But first we need to convert it to the form 0/0 or ∞/∞.

We do this by taking $\ln y$ where $y = x^{2x}$.

We get $\ln y = 2x\ln(x) = 2\dfrac{\ln(x)}{1/x}$.

Thus $x \ln x$ is of the form ∞/∞ as $x \to 0^+$.

Using L'Hospital's rule we get $Lim_{x\to 0^+}\dfrac{\ln(x)}{1/x} = Lim_{x\to 0^+}\dfrac{(\ln(x))'}{(1/x)'} = Lim_{x\to 0^+}\dfrac{1/x}{(-1/x^2)}$

$= Lim_{x\to 0^+} -x = 0$.

So $\ln y = 2(x\ln x) \to 2(0) = 0$ and thus $y \to e^0 = 1$.

So the limit should be 1 in order to make the function continuous at 0.

3. (40 points]) The following involve the function $f(x) = 2 + \sqrt{1-t^2}$:

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   (b)Express the integral as an area and then use geometry to find area. Compare with answer from (a).

   Solution:

   (a) Each of the values given is the value of f at the midpoint of the intervals of length 0.2 given by (0,0.2), (0.2,0.4),...,(0.8,1.0).

   The approximation is $f(0.1)(0.2) + f(0.3)(0.2) + ... + f(0.9)(0.2) = 13.965 \times 0.2 = 2.793$ approximately.

   (b) Separating the integral into two we get $\int_0^1 f(t)\ dt = \int_0^1 2\ dt + \int_0^1 \sqrt{1-t^2}\ dt$

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   If E and r are fixed but R varies over all positive real numbers,

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$$P'(R) = E^2(R/(R+2r)^2)' = E^2\frac{1(R+2r)^2 - R(2(R+2r))}{(R+2r)^4}$$

$$= E^2\frac{R^2 + 4Rr + 4r^2 - 2R^2 - 4Rr}{(R+2r)^4} = E^2\frac{4r^2 - R^2}{(R+2r)^4}.$$

   $P'(R) = 0 \implies 4r^2 - R^2 = 0$. So we have $4r^2 = R^2 \implies 2r = R$ where we take only positive root because neither r nor R are negative.

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When $R = 0$ the power is also 0 and when $R \to \infty$ the power goes to 0 because the degree of the denominator is smaller and we see that $\dfrac{R}{(R+2r)^2} \to \dfrac{R/R^2}{(R+2r)^2/R^2} \to \dfrac{1/R}{(1+\frac{2r}{R})^2} \to 0$. (divided above and below by highest power, namely R^2). You can also do this using L'Hospital's rule.

We see that at the boundaries $P \to 0$. So $R = 2r$ is also the absolute maximum and the maximum value of the power when $R = 2r$ is $E^2(2r)/(2r+2r)^2 = E^2(2r)/(4r)^2 = E^2/8r$.

5. (extra credit 20 points) Find the area under the curve $y = \dfrac{2x}{1+2x^2}$ between $x = 0$ and $x = 1$. You must use integration by substitution. Your answer will be $(\ln 3)/2$. (corrected)

   Solution:

   Let $u = 1 + 2x^2$. Then $du = 4xdx., \; so \; du/2 = 2xdx$ Also $u(0) = 1, u(1) = 3$.

$$\text{Area } = \int_0^1 \frac{2x\ dx}{1+x^2} = \int_1^3 \frac{du/2}{u} = \frac{1}{2}\left[\ln u\right]_1^3 = (\ln 3 - \ln 1)/2 = (\ln 3)/2.$$

   We didn't have to write $\ln |u|$ because $1 + 2x^2$ is always positive.